# ROBIN-TO-ROBIN MAPS AND KREIN-TYPE RESOLVENT FORMULAS FOR SCHRÖDINGER OPERATORS ON BOUNDED LIPSCHITZ DOMAINS

FRITZ GESZTESY AND MARIUS MITREA

*Dedicated to the memory of M. G. Krein (1907–1989).*

Abstract. We study Robin-to-Robin maps, and Krein-type resolvent formulas for Schrödinger operators on bounded Lipschitz domains in $\mathbb{R}^n$, $n \geqslant 2$, with generalized Robin boundary conditions.

## 1. Introduction

This paper is a direct continuation of our recent paper [35] in which we studied Schrödinger operators on bounded Lipschitz and $C^{1,r}$-domains with generalized Robin boundary conditions and discussed associated Robin-to-Dirichlet maps and Krein-type resolvent formulas. The paper [35], in turn, was a continuation of the earlier papers [32] and [36], where we studied general, not necessarily self-adjoint, Schrödinger operators on $C^{1,r}$-domains $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geqslant 2$, with compact boundaries $\partial\Omega$, $(1/2) < r < 1$ (including unbounded domains, i.e., exterior domains) with Dirichlet and Neumann boundary conditions on $\partial\Omega$. Our results also applied to convex domains $\Omega$ and to domains satisfying a uniform exterior ball condition. In addition, a careful discussion of locally singular potentials $V$ with close to optimal local behavior of $V$ was provided in [32] and [36].

In the current paper and in [35], we are exploring a different direction: Rather than discussing potentials with close to optimal local behavior, we will assume that $V \in L^\infty(\Omega; d^n x)$ and hence essentially replace it by zero nearly everywhere in this paper. On the other hand, instead of treating Dirichlet and Neumann boundary conditions at $\partial\Omega$, we now consider generalized Robin and again Dirichlet boundary conditions, but under minimal smoothness conditions on the domain $\Omega$, that is, we now consider Lipschitz domains $\Omega$. Additionally, to reduce some technicalities, we will assume that $\Omega$ is bounded throughout this paper. The principal new result in this paper is a derivation of Krein-type resolvent formulas for Schrödinger operators on bounded Lipschitz domains $\Omega$ in connection with two different generalized Robin boundary conditions on $\partial\Omega$.

In Section 2 we recall our recent detailed discussion of self-adjoint Laplacians with generalized Robin (and Dirichlet) boundary conditions on $\partial\Omega$ in [35]. In Section 3 we summarize generalized Robin and Dirichlet boundary value problems and introduce associated Robin-to-Dirichlet and Dirichlet-to-Robin maps following [35]. Section 4 is devoted to Krein-type resolvent formulas connecting Dirichlet and generalized Robin Laplacians with the help of the Robin-to-Dirichlet map. Section 5 contains our principal new results and studies Robin-to-Robin maps and general Krein-type formulas involving Robin-to-Robin maps. Appendix A collects useful material on Sobolev spaces

---

*Date*: May 16, 2008.

2000 *Mathematics Subject Classification.* Primary: 35J10, 35J25, 35Q40; Secondary: 35P05, 47A10, 47F05.

*Key words and phrases.* Multi-dimensional Schrödinger operators, bounded Lipschitz domains, Robin-to-Dirichlet and Dirichlet-to-Neumann maps.

Based upon work partially supported by the US National Science Foundation under Grant Nos. DMS-0400639 and FRG-0456306.

and trace maps for Lipschitz domains. Appendix B summarizes pertinent facts on sesquilinear forms and their associated linear operators.

While we formulate and prove all results in this paper for self-adjoint generalized Robin Laplacians and Dirichlet Laplacians, we emphasize that all results in this paper immediately extend to closed Schrödinger operators $H_{\Theta,\Omega} = -\Delta_{\Theta,\Omega} + V$, $\operatorname{dom}\big(H_{\Theta,\Omega}\big) = \operatorname{dom}\big(-\Delta_{\Theta,\Omega}\big)$ in $L^2(\Omega; d^n x)$ for (not necessarily real-valued) potentials $V$ satisfying $V \in L^\infty(\Omega; d^n x)$, by consistently replacing $-\Delta$ by $-\Delta + V$, etc. More generally, all results extend directly to Kato–Rellich bounded potentials $V$ relative to $-\Delta_{\Theta,\Omega}$ with bound less than one.

Next, we briefly list most of the notational conventions used throughout this paper. Let $\mathcal{H}$ be a separable complex Hilbert space, $(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$ the scalar product in $\mathcal{H}$ (linear in the second factor), and $I_{\mathcal{H}}$ the identity operator in $\mathcal{H}$. Next, let $T$ be a linear operator mapping (a subspace of) a Banach space into another, with $\operatorname{dom}(T)$ and $\operatorname{ran}(T)$ denoting the domain and range of $T$. The spectrum (resp., essential spectrum) of a closed linear operator in $\mathcal{H}$ will be denoted by $\sigma(\,\cdot\,)$ (resp., $\sigma_{\rm ess}(\,\cdot\,)$). The Banach spaces of bounded and compact linear operators in $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_\infty(\mathcal{H})$, respectively. Similarly, $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{B}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ will be used for bounded and compact operators between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Moreover, $\mathcal{X}_1 \hookrightarrow \mathcal{X}_2$ denotes the continuous embedding of the Banach space $\mathcal{X}_1$ into the Banach space $\mathcal{X}_2$. Throughout this manuscript, if $X$ denotes a Banach space, $X^*$ denotes the *adjoint space* of continuous conjugate linear functionals on $X$, that is, the *conjugate dual space* of $X$ (rather than the usual dual space of continuous linear functionals on $X$). This avoids the well-known awkward distinction between adjoint operators in Banach and Hilbert spaces (cf., e.g., the pertinent discussion in [29, p. 3–4]).

Finally, a notational comment: For obvious reasons in connection with quantum mechanical applications, we will, with a slight abuse of notation, dub $-\Delta$ (rather than $\Delta$) as the "Laplacian" in this paper.

## 2. Laplace Operators with Generalized Robin Boundary Conditions

In this section we recall various properties of general Laplacians $-\Delta_{\Theta,\Omega}$ in $L^2(\Omega; d^n x)$ including Dirichlet, $-\Delta_{D,\Omega}$, and Neumann, $-\Delta_{N,\Omega}$, Laplacians, generalized Robin-type Laplacians, and Laplacians corresponding to classical Robin boundary conditions associated with bounded open Lipschitz domains. For details we refer to our recent paper [35].

We start with introducing our precise assumptions on the set $\Omega$ and the boundary operator $\Theta$ which subsequently will be employed in defining the boundary condition on $\partial\Omega$:

**Hypothesis 2.1.** *Let $n \in \mathbb{N}$, $n \geq 2$, and assume that $\Omega \subset \mathbb{R}^n$ is an open, bounded, nonempty Lipschitz domain.*

We refer to Appendix A for more details on Lipschitz domains.

For simplicity of notation we will denote the identity operators in $L^2(\Omega; d^n x)$ and $L^2(\partial\Omega; d^{n-1}\omega)$ by $I_\Omega$ and $I_{\partial\Omega}$, respectively. In addition, we refer to Appendix A for our notation in connection with Sobolev spaces.

**Hypothesis 2.2.** *Assume Hypothesis* 2.1 *and suppose that $a_\Theta$ is a closed sesquilinear form in $L^2(\partial\Omega; d^{n-1}\omega)$ with domain $H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$, bounded from below by $c_\Theta \in \mathbb{R}$ (hence, in particular, $a_\Theta$ is symmetric). Denote by $\Theta \geqslant c_\Theta I_{\partial\Omega}$ the self-adjoint operator in $L^2(\partial\Omega; d^{n-1}\omega)$ uniquely associated with $a_\Theta$ (cf.* (B.27)) *and by $\widetilde{\Theta} \in \mathcal{B}\big(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\big)$ the extension of $\Theta$ as discussed in* (B.26) *and* (B.32).

Thus one has

$$\big\langle f, \widetilde{\Theta}\, g \big\rangle_{1/2} = \overline{\big\langle g, \widetilde{\Theta}\, f \big\rangle_{1/2}}, \quad f, g \in H^{1/2}(\partial\Omega). \tag{2.1}$$

$$\big\langle f, \widetilde{\Theta} f\big\rangle_{1/2} \geq c_\Theta \|f\|^2_{L^2(\partial\Omega;d^{n-1}\omega)}, \quad f \in H^{1/2}(\partial\Omega). \tag{2.2}$$

Here the sesquilinear form

$$\langle\,\cdot\,,\,\cdot\,\rangle_s = {}_{H^s(\partial\Omega)}\langle\,\cdot\,,\,\cdot\,\rangle_{H^{-s}(\partial\Omega)} \colon H^s(\partial\Omega) \times H^{-s}(\partial\Omega) \to \mathbb{C}, \quad s \in [0,1], \tag{2.3}$$

(antilinear in the first, linear in the second factor), denotes the duality pairing between $H^s(\partial\Omega)$ and

$$H^{-s}(\partial\Omega) = \big(H^s(\partial\Omega)\big)^*, \quad s \in [0,1], \tag{2.4}$$

such that

$$\langle f, g\rangle_s = \int_{\partial\Omega} d^{n-1}\omega(\xi)\, \overline{f(\xi)} g(\xi), \quad f \in H^s(\partial\Omega),\ g \in L^2(\partial\Omega; d^{n-1}\omega) \hookrightarrow H^{-s}(\partial\Omega),\ s \in [0,1], \tag{2.5}$$

and $d^{n-1}\omega$ denotes the surface measure on $\partial\Omega$.

Hypothesis 2.1 on $\Omega$ is used throughout this paper. Similarly, Hypothesis 2.2 is assumed whenever the boundary operator $\widetilde{\Theta}$ is involved. (Later in this section, and the next, we will occasionally strengthen our hypotheses.)

We introduce the boundary trace operator $\gamma_D^0$ (the Dirichlet trace) by

$$\gamma_D^0 \colon C(\overline{\Omega}) \to C(\partial\Omega), \quad \gamma_D^0 u = u|_{\partial\Omega}. \tag{2.6}$$

Then there exists a bounded, linear operator $\gamma_D$ (cf., e.g., [58, Theorem 3.38]),

$$\begin{aligned} &\gamma_D \colon H^s(\Omega) \to H^{s-(1/2)}(\partial\Omega) \hookrightarrow L^2(\partial\Omega; d^{n-1}\omega), \quad 1/2 < s < 3/2, \\ &\gamma_D \colon H^{3/2}(\Omega) \to H^{1-\varepsilon}(\partial\Omega) \hookrightarrow L^2(\partial\Omega; d^{n-1}\omega), \quad \varepsilon \in (0,1), \end{aligned} \tag{2.7}$$

whose action is compatible with that of $\gamma_D^0$. That is, the two Dirichlet trace operators coincide on the intersection of their domains. Moreover, we recall that

$$\gamma_D \colon H^s(\Omega) \to H^{s-(1/2)}(\partial\Omega) \text{ is onto for } 1/2 < s < 3/2. \tag{2.8}$$

While, in the class of bounded Lipschitz subdomains in $\mathbb{R}^n$, the end-point cases $s = 1/2$ and $s = 3/2$ of $\gamma_D \in \mathcal{B}\big(H^s(\Omega), H^{s-(1/2)}(\partial\Omega)\big)$ fail, we nonetheless have

$$\gamma_D \in \mathcal{B}\big(H^{(3/2)+\varepsilon}(\Omega), H^1(\partial\Omega)\big), \quad \varepsilon > 0. \tag{2.9}$$

See Lemma A.2 for a proof. Below we augment this with the following result:

**Lemma 2.3.** *Assume Hypothesis* 2.1. *Then for each $s > -3/2$, the restriction to boundary operator* (2.6) *extends to a linear operator*

$$\gamma_D : \big\{u \in H^{1/2}(\Omega) \,\big|\, \Delta u \in H^s(\Omega)\big\} \to L^2(\partial\Omega; d^{n-1}\omega), \tag{2.10}$$

*is compatible with* (2.7), *and is bounded when $\{u \in H^{1/2}(\Omega) \,|\, \Delta u \in H^s(\Omega)\}$ is equipped with the natural graph norm $u \mapsto \|u\|_{H^{1/2}(\Omega)} + \|\Delta u\|_{H^s(\Omega)}$.*

*Furthermore, for each $s > -3/2$, the restriction to boundary operator* (2.6) *also extends to a linear operator*

$$\gamma_D : \big\{u \in H^{3/2}(\Omega) \,\big|\, \Delta u \in H^{1+s}(\Omega)\big\} \to H^1(\partial\Omega), \tag{2.11}$$

*which, once again, is compatible with* (2.7), *and is bounded when $\{u \in H^{3/2}(\Omega) \,|\, \Delta u \in H^{1+s}(\Omega)\}$ is equipped with the natural graph norm $u \mapsto \|u\|_{H^{3/2}(\Omega)} + \|\Delta u\|_{H^{1+s}(\Omega)}$.*

Next, we introduce the operator $\gamma_N$ (the Neumann trace) by

$$\gamma_N = \nu \cdot \gamma_D \nabla \colon H^{s+1}(\Omega) \to L^2(\partial\Omega; d^{n-1}\omega), \quad 1/2 < s < 3/2, \tag{2.12}$$

where $\nu$ denotes the outward pointing normal unit vector to $\partial\Omega$. It follows from (2.7) that $\gamma_N$ is also a bounded operator. We wish to further extend the action of the Neumann trace operator (2.12) to other (related) settings. To set the stage, assume Hypothesis 2.1 and recall that the inclusion

$$\iota : H^s(\Omega) \hookrightarrow \big(H^1(\Omega)\big)^*, \quad s > -1/2, \tag{2.13}$$

is well-defined and bounded. Then, we introduce the weak Neumann trace operator

$$\widetilde\gamma_N \colon \big\{u \in H^1(\Omega) \,\big|\, \Delta u \in H^s(\Omega)\big\} \to H^{-1/2}(\partial\Omega), \quad s > -1/2, \tag{2.14}$$

as follows: Given $u \in H^1(\Omega)$ with $\Delta u \in H^s(\Omega)$ for some $s > -1/2$, we set (with $\iota$ as in (2.13))

$$\langle \phi, \widetilde\gamma_N u\rangle_{1/2} = \int_\Omega d^n x\, \overline{\nabla\Phi(x)} \cdot \nabla u(x) + {}_{H^1(\Omega)}\langle \Phi, \iota(\Delta u)\rangle_{(H^1(\Omega))^*}, \tag{2.15}$$

for all $\phi \in H^{1/2}(\partial\Omega)$ and $\Phi \in H^1(\Omega)$ such that $\gamma_D\Phi = \phi$. We note that this definition is independent of the particular extension $\Phi$ of $\phi$, and that $\widetilde\gamma_N$ is a bounded extension of the Neumann trace operator $\gamma_N$ defined in (2.12). As was the case of the Dirichlet trace, the (weak) Neumann trace operator (2.14), (2.15) is onto (cf. [35]). For additional details we refer to equations (A.8)–(A.10). Next, we wish to discuss the end-point case $s = 1/2$ of (2.12).

**Lemma 2.4.** *Assume Hypothesis* 2.1*. Then the Neumann trace operator* (2.12) *also extends to*

$$\widetilde\gamma_N : \big\{u \in H^{3/2}(\Omega) \,\big|\, \Delta u \in L^2(\Omega; d^n x)\big\} \to L^2(\partial\Omega; d^{n-1}\omega) \tag{2.16}$$

*in a bounded fashion when the space* $\big\{u \in H^{3/2}(\Omega) \,|\, \Delta u \in L^2(\Omega; d^n x)\big\}$ *is equipped with the natural graph norm* $u \mapsto \|u\|_{H^{3/2}(\Omega)} + \|\Delta u\|_{L^2(\Omega;d^n x)}$*. This extension is compatible with* (2.14)*.*

For future purposes, we shall need yet another extension of the concept of Neumann trace. This requires some preparations (throughout, Hypothesis 2.1 is enforced). First, we recall that, as is well-known (see, e.g., [40]), one has the natural identification

$$\big(H^1(\Omega)\big)^* \equiv \big\{u \in H^{-1}(\mathbb{R}^n) \,\big|\, \mathrm{supp}\,(u) \subseteq \overline{\Omega}\big\}. \tag{2.17}$$

Note that the latter is a closed subspace of $H^{-1}(\mathbb{R}^n)$. In particular, if $R_\Omega u = u|_\Omega$ denotes the operator of restriction to $\Omega$ (considered in the sense of distributions), then

$$R_\Omega : \big(H^1(\Omega)\big)^* \to H^{-1}(\Omega) \tag{2.18}$$

is well-defined, linear and bounded. Furthermore, the composition of $R_\Omega$ in (2.18) with $\iota$ in (2.13) is the natural inclusion of $H^s(\Omega)$ into $H^{-1}(\Omega)$. Next, given $z \in \mathbb{C}$, set

$$W_z(\Omega) = \big\{(u, f) \in H^1(\Omega) \times \big(H^1(\Omega)\big)^* \,\big|\, (-\Delta - z)u = f|_\Omega \text{ in } \mathcal{D}'(\Omega)\big\}, \tag{2.19}$$

equipped with the norm inherited from $H^1(\Omega) \times \big(H^1(\Omega)\big)^*$. We then denote by

$$\widetilde\gamma_{\mathcal{N}} \colon W_z(\Omega) \to H^{-1/2}(\partial\Omega) \tag{2.20}$$

the *ultra weak* Neumann trace operator defined by

$$\begin{aligned} \langle \phi, \widetilde\gamma_{\mathcal{N}}(u, f)\rangle_{1/2} &= \int_\Omega d^n x\, \overline{\nabla\Phi(x)} \cdot \nabla u(x) \\ &\quad - z \int_\Omega d^n x\, \overline{\Phi(x)} u(x) - {}_{H^1(\Omega)}\langle \Phi, f\rangle_{(H^1(\Omega))^*}, \quad (u, f) \in W_z(\Omega), \end{aligned} \tag{2.21}$$

for all $\phi \in H^{1/2}(\partial\Omega)$ and $\Phi \in H^1(\Omega)$ such that $\gamma_D\Phi = \phi$. Once again, this definition is independent of the particular extension $\Phi$ of $\phi$. Also, as was the case of the Dirichlet trace, the ultra weak Neumann trace operator (2.20), (2.21) is onto (this is a corollary of Theorem 4.4). For additional details we refer to equations (A.8)–(A.10).

The relationship between the ultra weak Neumann trace operator (2.20), (2.21) and the weak Neumann trace operator (2.14), (2.15) can be described as follows. Given $s > -1/2$ and $z \in \mathbb{C}$, denote by

$$j_z : \{u \in H^1(\Omega) \,|\, \Delta u \in H^s(\Omega)\} \to W_z(\Omega) \tag{2.22}$$

the injection

$$j_z(u) = (u, \iota(-\Delta u - zu)), \quad u \in H^1(\Omega),\ \Delta u \in H^s(\Omega), \tag{2.23}$$

where $\iota$ is as in (2.13). Then

$$\widetilde{\gamma}_{\mathcal{N}} \circ j_z = \widetilde{\gamma}_N. \tag{2.24}$$

Thus, from this perspective, $\widetilde{\gamma}_{\mathcal{N}}$ can also be regarded as a bounded extension of the Neumann trace operator $\gamma_N$ defined in (2.12).

Moving on, we shall now describe a family of self-adjoint Laplace operators $-\Delta_{\Theta,\Omega}$ in $L^2(\Omega; d^n x)$ indexed by the boundary operator $\Theta$. We will refer to $-\Delta_{\Theta,\Omega}$ as the generalized Robin Laplacian.

**Theorem 2.5.** *Assume Hypothesis* 2.2*. Then the generalized Robin Laplacian, $-\Delta_{\Theta,\Omega}$, defined by*

$$-\Delta_{\Theta,\Omega} = -\Delta, \quad \operatorname{dom}(-\Delta_{\Theta,\Omega}) = \big\{u \in H^1(\Omega) \,\big|\, \Delta u \in L^2(\Omega; d^n x);\ \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u = 0 \text{ in } H^{-1/2}(\partial\Omega)\big\}, \tag{2.25}$$

*is self-adjoint and bounded from below in $L^2(\Omega; d^n x)$. Moreover,*

$$\operatorname{dom}\big(|-\Delta_{\Theta,\Omega}|^{1/2}\big) = H^1(\Omega). \tag{2.26}$$

*In addition, $-\Delta_{\Theta,\Omega}$, has purely discrete spectrum bounded from below, in particular,*

$$\sigma_{\rm ess}(-\Delta_{\Theta,\Omega}) = \emptyset. \tag{2.27}$$

The important special case where $\Theta$ corresponds to the operator of multiplication by a real-valued, essentially bounded function $\theta$ leads to Robin boundary conditions we discuss next:

**Corollary 2.6.** *In addition to Hypothesis* 2.1*, assume that $\Theta$ is the operator of multiplication in $L^2(\partial\Omega; d^{n-1}\omega)$ by the real-valued function $\theta$ satisfying $\theta \in L^\infty(\partial\Omega; d^{n-1}\omega)$. Then $\Theta$ satisfies the conditions in Hypothesis* 2.2 *resulting in the self-adjoint and bounded from below Laplacian $-\Delta_{\theta,\Omega}$ in $L^2(\Omega; d^n x)$ with Robin boundary conditions on $\partial\Omega$ in* (2.25) *given by*

$$(\widetilde{\gamma}_N + \theta\gamma_D)u = 0 \ \text{ in } \ H^{-1/2}(\partial\Omega). \tag{2.28}$$

**Remark 2.7.** $(i)$ In the case of a smooth boundary $\partial\Omega$, the boundary conditions in (2.28) are also called "classical" boundary conditions (cf., e.g., [73]); in the more general case of bounded Lipschitz domains we also refer to [5] and [80, Ch. 4] in this context. Next, we point out that, in [51], the authors have dealt with the case of Laplace operators in bounded Lipschitz domains, equipped with local boundary conditions of Robin-type, with boundary data in $L^p(\partial\Omega; d^{n-1}\omega)$, and produced nontangential maximal function estimates. For the case $p = 2$, when our setting agrees with that of [51], some of our results in this section and the following are a refinement of those in [51]. Maximal $L^p$-regularity and analytic contraction semigroups of Dirichlet and Neumann Laplacians on bounded Lipschitz domains were studied in [83]. Holomorphic $C_0$-semigroups of the Laplacian with Robin boundary conditions on bounded Lipschitz domains have been discussed in [81]. Moreover, Robin boundary conditions for elliptic boundary value problems on arbitrary open domains were first studied by Maz'ya [56], [57, Sect. 4.11.6], and subsequently in [22] (see also [23] which treats the case of the Laplacian). In addition, Robin-type boundary conditions involving measures on the boundary for very general domains $\Omega$ were intensively discussed in terms of quadratic forms and capacity methods in the literature, and we refer, for instance, to [5], [6], [15], [80], and the references therein.

(*ii*) In the special case $\theta = 0$ (resp., $\widetilde{\Theta} = 0$), that is, in the case of the Neumann Laplacian, we will also use the notation

$$-\Delta_{N,\Omega} = -\Delta_{0,\Omega}. \tag{2.29}$$

The case of the Dirichlet Laplacian $-\Delta_{D,\Omega}$ associated with $\Omega$ formally corresponds to $\Theta = \infty$ and so we recall its treatment in the next result. To state it, recall that, given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$,

$$H_0^1(\Omega) = \{u \in H^1(\Omega) \,|\, \gamma_D u = 0 \text{ on } \partial\Omega\}. \tag{2.30}$$

**Theorem 2.8.** *Assume Hypothesis* 2.1. *Then the Dirichlet Laplacian, $-\Delta_{D,\Omega}$, defined by*

$$\begin{aligned} -\Delta_{D,\Omega} = -\Delta, \quad \operatorname{dom}(-\Delta_{D,\Omega}) &= \big\{u \in H^1(\Omega) \,\big|\, \Delta u \in L^2(\Omega; d^n x);\ \gamma_D u = 0 \text{ in } H^{1/2}(\partial\Omega)\big\} \\ &= \big\{u \in H_0^1(\Omega) \,\big|\, \Delta u \in L^2(\Omega; d^n x)\big\}, \end{aligned} \tag{2.31}$$

*is self-adjoint and strictly positive in $L^2(\Omega; d^n x)$. Moreover,*

$$\operatorname{dom}\big((-\Delta_{D,\Omega})^{1/2}\big) = H_0^1(\Omega). \tag{2.32}$$

Since $\Omega$ is open and bounded, it is well-known that $-\Delta_{D,\Omega}$ has purely discrete spectrum contained in $(0,\infty)$, in particular, $\sigma_{\rm ess}(-\Delta_{D,\Omega}) = \emptyset$.

## 3. Generalized Robin and Dirichlet Boundary Value Problems and Robin-to-Dirichlet and Dirichlet-to-Robin Maps

This section is devoted to generalized Robin and Dirichlet boundary value problems associated with the Helmholtz differential expression $-\Delta - z$ in connection with the open set $\Omega$. In addition, we provide a detailed discussion of Robin-to-Dirichlet maps, $M_{\Theta,D,\Omega}^{(0)}$, in $L^2(\partial\Omega; d^{n-1}\omega)$. Again, the material in this section is taken from [35, Sect. 3].

In this section we strengthen Hypothesis 2.2 by adding assumption (3.1) below:

**Hypothesis 3.1.** *In addition to Hypothesis* 2.2 *suppose that*

$$\widetilde{\Theta} \in \mathcal{B}_\infty\big(H^1(\partial\Omega), L^2(\partial\Omega; d^{n-1}\omega)\big). \tag{3.1}$$

We note that (3.1) is satisfied whenever there exists some $\varepsilon > 0$ such that

$$\widetilde{\Theta} \in \mathcal{B}\big(H^{1-\varepsilon}(\partial\Omega), L^2(\partial\Omega; d^{n-1}\omega)\big). \tag{3.2}$$

We recall the definition of the weak Neumann trace operator $\widetilde{\gamma}_N$ in (2.14), (2.15) and start with the Helmholtz Robin boundary value problems:

**Theorem 3.2.** *Assume Hypothesis* 3.1 *and suppose that $z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega})$. Then for every $g \in L^2(\partial\Omega; d^{n-1}\omega)$, the following generalized Robin boundary value problem,*

$$\begin{cases} (-\Delta - z)u = 0 \text{ in } \Omega, \quad u \in H^{3/2}(\Omega), \\ \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u = g \text{ on } \partial\Omega, \end{cases} \tag{3.3}$$

*has a unique solution $u = u_\Theta$. This solution $u_\Theta$ satisfies*

$$\begin{aligned} &\gamma_D u_\Theta \in H^1(\partial\Omega), \quad \widetilde{\gamma}_N u_\Theta \in L^2(\partial\Omega; d^{n-1}\omega), \\ &\|\gamma_D u_\Theta\|_{H^1(\partial\Omega)} + \|\widetilde{\gamma}_N u_\Theta\|_{L^2(\partial\Omega; d^{n-1}\omega)}, \leq C\|g\|_{L^2(\partial\Omega; d^{n-1}\omega)} \end{aligned} \tag{3.4}$$

*and*

$$\|u_\Theta\|_{H^{3/2}(\Omega)} \leq C\|g\|_{L^2(\partial\Omega; d^{n-1}\omega)}, \tag{3.5}$$

*for some constant constant $C = C(\Theta, \Omega, z) > 0$. Finally,*

$$\big[\gamma_D(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\omega), H^{3/2}(\Omega)\big), \tag{3.6}$$

*and the solution $u_\Theta$ is given by the formula*

$$u_\Theta = \big(\gamma_D(-\Delta_{\Theta,\Omega} - \overline{z} I_\Omega)^{-1}\big)^* g. \tag{3.7}$$

Next, we turn to the Dirichlet case originally treated in [36, Theorem 3.1] but under stronger regularity conditions on $\Omega$.

**Theorem 3.3.** *Assume Hypothesis* 2.1 *and suppose that $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$. Then for every $f \in H^1(\partial\Omega)$, the following Dirichlet boundary value problem,*

$$\begin{cases} (-\Delta - z)u = 0 \ \text{in} \ \Omega, \quad u \in H^{3/2}(\Omega), \\ \gamma_D u = f \ \text{on} \ \partial\Omega, \end{cases} \tag{3.8}$$

*has a unique solution $u = u_D$. This solution $u_D$ satisfies*

$$\widetilde{\gamma}_N u_D \in L^2(\partial\Omega; d^{n-1}\omega) \quad \text{and} \quad \|\widetilde{\gamma}_N u_D\|_{L^2(\partial\Omega; d^{n-1}\omega)} \le C_D \|f\|_{H^1(\partial\Omega)}, \tag{3.9}$$

*for some constant $C_D = C_D(\Omega, z) > 0$. Moreover,*

$$\|u_D\|_{H^{3/2}(\Omega)} \le C_D \|f\|_{H^1(\partial\Omega)}. \tag{3.10}$$

*Finally,*

$$\big[\widetilde{\gamma}_N(-\Delta_{D,\Omega} - \overline{z} I_\Omega)^{-1}\big]^* \in \mathcal{B}\big(H^1(\partial\Omega), H^{3/2}(\Omega)\big), \tag{3.11}$$

*and the solution $u_D$ is given by the formula*

$$u_D = -\big(\widetilde{\gamma}_N(-\Delta_{D,\Omega} - \overline{z} I_\Omega)^{-1}\big)^* f. \tag{3.12}$$

In addition to Theorem 3.3, we recall the following result.

**Lemma 3.4.** *Assume Hypothesis* 2.1 *and suppose that $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$. Then*

$$\widetilde{\gamma}_N(-\Delta_{D,\Omega} - z I_\Omega)^{-1} \in \mathcal{B}\big(L^2(\Omega; d^n x), L^2(\partial\Omega; d^{n-1}\omega)\big), \tag{3.13}$$

*and*

$$\big[\widetilde{\gamma}_N(-\Delta_{D,\Omega} - z I_\Omega)^{-1}\big]^* \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\omega), L^2(\Omega; d^n x)\big). \tag{3.14}$$

Assuming Hypothesis 3.1, we now introduce the Dirichlet-to-Robin map $M_{D,\Theta,\Omega}^{(0)}(z)$ associated with $(-\Delta - z)$ on $\Omega$, as follows,

$$M_{D,\Theta,\Omega}^{(0)}(z) \colon \begin{cases} H^1(\partial\Omega) \to L^2(\partial\Omega; d^{n-1}\omega), \\ \qquad f \mapsto -\big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big) u_D, \end{cases} \qquad z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega}), \tag{3.15}$$

where $u_D$ is the unique solution of

$$(-\Delta - z)u = 0 \ \text{in} \ \Omega, \quad u \in H^{3/2}(\Omega), \quad \gamma_D u = f \ \text{on} \ \partial\Omega. \tag{3.16}$$

Continuing to assume Hypothesis 3.1, we next introduce the Robin-to-Dirichlet map $M_{\Theta,D,\Omega}^{(0)}(z)$ associated with $(-\Delta - z)$ on $\Omega$, as follows,

$$M_{\Theta,D,\Omega}^{(0)}(z) \colon \begin{cases} L^2(\partial\Omega; d^{n-1}\omega) \to H^1(\partial\Omega), \\ \qquad g \mapsto \gamma_D u_\Theta, \end{cases} \qquad z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega}), \tag{3.17}$$

where $u_\Theta$ is the unique solution of

$$(-\Delta - z)u = 0 \ \text{in} \ \Omega, \quad u \in H^{3/2}(\Omega), \quad \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u = g \ \text{on} \ \partial\Omega. \tag{3.18}$$

We note that Robin-to-Dirichlet maps have also been studied in [9].

Next we recall one of the main results in [35]:

**Theorem 3.5.** *Assume Hypothesis* 3.1. *Then*

$$M_{D,\Theta,\Omega}^{(0)}(z) \in \mathcal{B}\big(H^1(\partial\Omega), L^2(\partial\Omega; d^{n-1}\omega)\big), \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega}), \tag{3.19}$$

*and*

$$M_{D,\Theta,\Omega}^{(0)}(z) = \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)\big[\big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}\big]^*, \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega}). \tag{3.20}$$

*Moreover,*

$$M_{\Theta,D,\Omega}^{(0)}(z) \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\omega), H^1(\partial\Omega)\big), \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega}), \tag{3.21}$$

*and, in fact,*

$$M_{\Theta,D,\Omega}^{(0)}(z) \in \mathcal{B}_\infty\big(L^2(\partial\Omega; d^{n-1}\omega)\big), \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega}). \tag{3.22}$$

*In addition,*

$$M_{\Theta,D,\Omega}^{(0)}(z) = \gamma_D\big[\gamma_D(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}\big]^*, \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega}). \tag{3.23}$$

*Finally, let $z \in \mathbb{C}\backslash(\sigma(-\Delta_{D,\Omega}) \cup \sigma(-\Delta_{\Theta,\Omega}))$. Then*

$$M_{\Theta,D,\Omega}^{(0)}(z) = -M_{D,\Theta,\Omega}^{(0)}(z)^{-1}. \tag{3.24}$$

**Remark 3.6.** In the above considerations, the special case $\Theta = 0$ represents the frequently studied Neumann-to-Dirichlet and Dirichlet-to-Neumann maps $M_{N,D,\Omega}^{(0)}(z)$ and $M_{D,N,\Omega}^{(0)}(z)$, respectively. That is, $M_{N,D,\Omega}^{(0)}(z) = M_{0,D,\Omega}^{(0)}(z)$ and $M_{D,N,\Omega}^{(0)}(z) = M_{D,0,\Omega}^{(0)}(z)$. Thus, as a corollary of Theorem 3.5 we have

$$M_{N,D,\Omega}^{(0)}(z) = -M_{D,N,\Omega}^{(0)}(z)^{-1}, \tag{3.25}$$

whenever Hypothesis 2.1 holds and $z \in \mathbb{C}\backslash(\sigma(-\Delta_{D,\Omega}) \cup \sigma(-\Delta_{N,\Omega}))$.

**Remark 3.7.** We emphasize again that all results in this section immediately extend to Schrödinger operators $H_{\Theta,\Omega} = -\Delta_{\Theta,\Omega} + V$, $\operatorname{dom}\big(H_{\Theta,\Omega}\big) = \operatorname{dom}\big(-\Delta_{\Theta,\Omega}\big)$ in $L^2(\Omega; d^n x)$ for (not necessarily real-valued) potentials $V$ satisfying $V \in L^\infty(\Omega; d^n x)$, or more generally, for potentials $V$ which are Kato–Rellich bounded with respect to $-\Delta_{\Theta,\Omega}$ with bound less than one. Denoting the corresponding $M$-operators by $M_{D,N,\Omega}(z)$ and $M_{\Theta,D,\Omega}(z)$, respectively, we note, in particular, that (3.15)–(3.24) extend replacing $-\Delta$ by $-\Delta + V$ and restricting $z \in \mathbb{C}$ appropriately.

Our discussion of Weyl–Titchmarsh operators follows the earlier papers [32] and [36]. For related literature on Weyl–Titchmarsh operators, relevant in the context of boundary value spaces (boundary triples, etc.), we refer, for instance, to [2], [4], [11], [12], [16]–[20], [25]– [28], [31], [34], [37, Ch. 3], [39, Ch. 13], [54], [55], [59], [64], [65], [68]–[71], [79].

## 4. Some Variants of Krein's Resolvent Formula Involving Robin-to-Dirichlet Maps

In this section we recall some of the principal new results in [35], viz., variants of Krein's formula for the difference of resolvents of generalized Robin Laplacians and Dirichlet Laplacians on bounded Lipschitz domains. For details on the material in this section we refer to [35].

We start by weakening Hypothesis 3.1 by using assumption (4.1) below:

**Hypothesis 4.1.** *In addition to Hypothesis* 2.2 *suppose that*

$$\widetilde{\Theta} \in \mathcal{B}_\infty\big(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\big). \tag{4.1}$$

We note that condition (4.1) is satisfied if there exists some $\varepsilon > 0$ such that

$$\widetilde{\Theta} \in \mathcal{B}\big(H^{1/2-\varepsilon}(\partial\Omega), H^{-1/2}(\partial\Omega)\big). \tag{4.2}$$

We wish to point out that Hypothesis 3.1 is indeed stronger than Hypothesis 4.1 since (3.1) implies, via duality and interpolation (cf. the discussion in [35]), that

$$\widetilde{\Theta} \in \mathcal{B}_\infty\big(H^s(\partial\Omega), H^{s-1}(\partial\Omega)\big), \quad 0 \le s \le 1. \tag{4.3}$$

In our next two results below (Theorems 4.2–4.4) we discuss the solvability of the Dirichlet and Robin boundary value problems with solution in the energy space $H^1(\Omega)$.

**Theorem 4.2.** *Assume Hypothesis* 4.1 *and suppose that $z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega})$. Then for every $g \in H^{-1/2}(\partial\Omega)$, the following generalized Robin boundary value problem,*

$$\begin{cases} (-\Delta - z)u = 0 \text{ in } \Omega, \quad u \in H^1(\Omega), \\ \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u = g \text{ on } \partial\Omega, \end{cases} \tag{4.4}$$

*has a unique solution $u = u_\Theta$. Moreover, there exists a constant $C = C(\Theta, \Omega, z) > 0$ such that*

$$\|u_\Theta\|_{H^1(\Omega)} \leq C\|g\|_{H^{-1/2}(\partial\Omega)}. \tag{4.5}$$

*In particular,*

$$\big[\gamma_D(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* \in \mathcal{B}\big(H^{-1/2}(\partial\Omega), H^1(\Omega)\big), \tag{4.6}$$

*and the solution $u_\Theta$ of* (4.4) *is once again given by formula* (3.7).

**Remark 4.3.** As a byproduct of Theorem 4.2 (with $\Theta = 0$) we obtain that the weak Neumann trace $\widetilde{\gamma}_N$ in (2.14), (2.15) is onto.

In the following we denote by $\widetilde{I}_\Omega$ the continuous inclusion (embedding) map of $H^1(\Omega)$ into $\big(H^1(\Omega)\big)^*$. By a slight abuse of notation, we also denote the continuous inclusion map of $H_0^1(\Omega)$ into $\big(H_0^1(\Omega)\big)^*$ by the same symbol $\widetilde{I}_\Omega$. We recall the ultra weak Neumann trace operator $\widetilde{\gamma}_{\mathcal{N}}$ from (2.20), (2.21). Finally, assuming Hypothesis 4.1, we denote by

$$-\widetilde{\Delta}_{\Theta,\Omega} \in \mathcal{B}\big(H^1(\Omega), \big(H^1(\Omega)\big)^*\big) \tag{4.7}$$

the extension of $-\Delta_{\Theta,\Omega}$ in accordance with (B.26). In particular,

$${}_{H^1(\Omega)}\langle u, -\widetilde{\Delta}_{\Theta,\Omega}v\rangle_{(H^1(\Omega))^*} = \int_\Omega d^n x\, \overline{\nabla u(x)} \cdot \nabla v(x) + \big\langle \gamma_D u, \widetilde{\Theta}\gamma_D v\big\rangle_{1/2}, \quad u, v \in H^1(\Omega), \tag{4.8}$$

and $-\Delta_{\Theta,\Omega}$ is the restriction of $-\widetilde{\Delta}_{\Theta,\Omega}$ to $L^2(\Omega; d^n x)$ (cf. (B.27)).

**Theorem 4.4.** *Assume Hypothesis* 4.1 *and suppose that $z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega})$. Then for every $w \in (H^1(\Omega))^*$, the following generalized inhomogeneous Robin problem,*

$$\begin{cases} (-\Delta - z)u = w|_\Omega \text{ in } \mathcal{D}'(\Omega), \quad u \in H^1(\Omega), \\ \widetilde{\gamma}_{\mathcal{N}}(u, w) + \widetilde{\Theta}\gamma_D u = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.9}$$

*has a unique solution $u = u_{\Theta,w}$. Moreover, there exists a constant $C = C(\Theta, \Omega, z) > 0$ such that*

$$\|u_{\Theta,w}\|_{H^1(\Omega)} \leq C\|w\|_{(H^1(\partial\Omega))^*}. \tag{4.10}$$

*In particular, the operator $(-\Delta_{\Theta,\Omega} - zI_\Omega)^{-1}$, $z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega})$, originally defined as a bounded operator on $L^2(\Omega; d^n x)$,*

$$(-\Delta_{\Theta,\Omega} - zI_\Omega)^{-1} \in \mathcal{B}\big(L^2(\Omega; d^n x)\big), \tag{4.11}$$

*can be extended to a mapping in $\mathcal{B}\big(\big(H^1(\Omega)\big)^*, H^1(\Omega)\big)$, which in fact coincides with*

$$\big(-\widetilde{\Delta}_{\Theta,\Omega} - z\widetilde{I}_\Omega\big)^{-1} \in \mathcal{B}\big(\big(H^1(\Omega)\big)^*, H^1(\Omega)\big). \tag{4.12}$$

**Remark 4.5.** In the context of Theorem 4.4, it is useful to observe that for any $w \in \big(H^1(\Omega)\big)^*$, the function $u = \big(-\widetilde{\Delta}_{\Theta,\Omega} - z\widetilde{I}_\Omega\big)^{-1} w \in H^1(\Omega)$ satisfies

$$(-\Delta - z)u = w|_\Omega \text{ in } \mathcal{D}'(\Omega), \tag{4.13}$$

where the restriction of $w$ to $\Omega$ is interpreted by regarding $w$ as a distribution in $H^{-1}(\Omega)$ (cf. (2.17)). Indeed, the identification (2.17) associates to a functional $w \in \big(H^1(\Omega)\big)^*$ the distribution $\widehat{w} = w \circ R_\Omega \in H^{-1}(\mathbb{R}^n) = \big(H^1(\mathbb{R}^n)\big)^*$ (which happens to be supported in $\overline{\Omega}$). Consequently, if for an arbitrary test function $\varphi \in C_0^\infty(\Omega)$ we denote by $\widetilde{\varphi} \in C_0^\infty(\mathbb{R}^n)$ the extension of $\varphi$ by zero outside $\Omega$, we then have

$$\begin{aligned}
{}_{\mathcal{D}(\Omega)}\langle \varphi, \widehat{w}|_\Omega \rangle_{\mathcal{D}'(\Omega)} &= {}_{\mathcal{D}(\mathbb{R}^n)}\langle \widetilde{\varphi}, \widehat{w} \rangle_{\mathcal{D}'(\mathbb{R}^n)} \\
&= {}_{H^1(\mathbb{R}^n)}\langle \widetilde{\varphi}, \widehat{w} \rangle_{(H^1(\mathbb{R}^n))^*} = {}_{H^1(\Omega)}\langle R_\Omega(\widetilde{\varphi}), w \rangle_{(H^1(\Omega))^*} \\
&= {}_{H^1(\Omega)}\langle \varphi, w \rangle_{(H^1(\Omega))^*} = {}_{H^1(\Omega)}\big\langle \varphi, \big(-\widetilde{\Delta}_{\Theta,\Omega} - z\widetilde{I}_\Omega\big)u \big\rangle_{(H^1(\Omega))^*} \\
&= {}_{H^1(\Omega)}\big\langle \varphi, \big(-\widetilde{\Delta}_{\Theta,\Omega}\big)u \big\rangle_{(H^1(\Omega))^*} - z\,(\varphi, u)_{L^2(\Omega;d^nx)} \\
&= (\nabla\varphi, \nabla u)_{(L^2(\Omega;d^nx))^n} - z\,(\varphi, u)_{L^2(\Omega;d^nx)} \\
&= ((-\Delta - \overline{z})\varphi, u)_{L^2(\Omega;d^nx)},
\end{aligned} \tag{4.14}$$

on account of (4.8). This justifies (4.13).

**Remark 4.6.** Similar (yet simpler) considerations also show that the operator $(-\Delta_{D,\Omega} - zI_\Omega)^{-1}$, $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$, originally defined as bounded operator on $L^2(\Omega; d^nx)$,

$$(-\Delta_{D,\Omega} - zI_\Omega)^{-1} \in \mathcal{B}\big(L^2(\Omega; d^nx)\big), \tag{4.15}$$

extends to a mapping

$$\big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big)^{-1} \in \mathcal{B}\big(H^{-1}(\Omega); H_0^1(\Omega)\big). \tag{4.16}$$

Here $-\widetilde{\Delta}_{D,\Omega} \in \mathcal{B}\big(H_0^1(\Omega), H^{-1}(\Omega)\big)$ is the extension of $-\Delta_{D,\Omega}$ in accordance with (B.26). Indeed, the Lax–Milgram lemma applies and yields that

$$\big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big) \colon H_0^1(\Omega) \to \big(H_0^1(\Omega)\big)^* = H^{-1}(\Omega) \tag{4.17}$$

is, in fact, an isomorphism whenever $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$.

**Corollary 4.7.** *Assume Hypothesis* 4.1 *and suppose that* $z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega})$. *Then the operator* $M_{\Theta,D,\Omega}^{(0)}(z) \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\omega)\big)$ *in* (3.17), (3.18) *extends* (*in a compatible manner*) *to*

$$\widetilde{M}_{\Theta,D,\Omega}^{(0)}(z) \in \mathcal{B}\big(H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)\big), \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega}). \tag{4.18}$$

*In addition,* $\widetilde{M}_{\Theta,D,\Omega}^{(0)}(z)$ *permits the representation*

$$\widetilde{M}_{\Theta,D,\Omega}^{(0)}(z) = \gamma_D\big(-\widetilde{\Delta}_{\Theta,\Omega} - z\widetilde{I}_\Omega\big)^{-1}\gamma_D^*, \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega}). \tag{4.19}$$

*The same applies to the adjoint* $M_{\Theta,D,\Omega}^{(0)}(z)^* \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\omega)\big)$ *of* $M_{\Theta,D,\Omega}^{(0)}(z)$, *resulting in the bounded extension* $\big(\widetilde{M}_{\Theta,D,\Omega}^{(0)}(z)\big)^* \in \mathcal{B}\big(H^{-1/2}(\partial\Omega),\, H^{1/2}(\partial\Omega)\big)$, $z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega})$.

**Lemma 4.8.** *Assume Hypothesis* 4.1 *and suppose that* $z \in \mathbb{C}\backslash(\sigma(-\Delta_{\Theta,\Omega}) \cup \sigma(-\Delta_{D,\Omega}))$. *Then the following resolvent relation holds on* $\big(H^1(\Omega)\big)^*$,

$$\begin{aligned}
\big(-\widetilde{\Delta}_{\Theta,\Omega} - z\widetilde{I}_\Omega\big)^{-1} &= \big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big)^{-1} \circ R_\Omega \\
&\quad + \big(-\widetilde{\Delta}_{\Theta,\Omega} - z\widetilde{I}_\Omega\big)^{-1}\gamma_D^*\widetilde{\gamma}_{\mathcal{N}}\big(\big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big)^{-1} \circ R_\Omega, I_{\mathbb{R}^n}\big).
\end{aligned} \tag{4.20}$$

We also recall the following regularity result for the Robin resolvent.

**Lemma 4.9.** *Assume Hypothesis* 3.1 *and suppose that* $z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega})$. *Then*

$$(-\Delta_{\Theta,\Omega} - zI_\Omega)^{-1} : L^2(\Omega; d^n x) \to \big\{u \in H^{3/2}(\Omega) \,\big|\, \Delta u \in L^2(\Omega; d^n x)\big\} \tag{4.21}$$

*is a well-defined bounded operator, where the space* $\big\{u \in H^{3/2}(\Omega) \,\big|\, \Delta u \in L^2(\Omega; d^n x)\big\}$ *is equipped with the natural graph norm* $u \mapsto \|u\|_{H^{3/2}(\Omega)} + \|\Delta u\|_{L^2(\Omega;d^n x)}$.

Under Hypothesis 4.1, (4.12) and (2.7) yield

$$\gamma_D\big(-\widetilde{\Delta}_{\Theta,\Omega} - z\widetilde{I}_\Omega\big)^{-1} \in \mathcal{B}\big((H^1(\Omega))^*, H^{1/2}(\partial\Omega)\big). \tag{4.22}$$

Hence, by duality,

$$\big[\gamma_D\big(-\widetilde{\Delta}_{\Theta,\Omega} - z\widetilde{I}_\Omega\big)^{-1}\big]^* \in \mathcal{B}\big(H^{-1/2}(\partial\Omega), H^1(\Omega)\big). \tag{4.23}$$

Next we complement this with the following result.

**Corollary 4.10.** *Assume Hypothesis* 3.1 *and suppose that* $z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega})$. *Then*

$$\gamma_D(-\Delta_{\Theta,\Omega} - zI_\Omega)^{-1} \in \mathcal{B}\big(L^2(\Omega; d^n x), H^1(\partial\Omega)\big). \tag{4.24}$$

*In particular,*

$$\big[\gamma_D(-\Delta_{\Theta,\Omega} - zI_\Omega)^{-1}\big]^* \in \mathcal{B}\big(H^{-1}(\partial\Omega), L^2(\Omega; d^n x)\big) \hookrightarrow \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\Omega), L^2(\Omega; d^n x)\big). \tag{4.25}$$

*In addition, the operator* (4.25) *is compatible with* (4.23) *in the sense that*

$$\big[\gamma_D(-\Delta_{\Theta,\Omega} - zI_\Omega)^{-1}\big]^* f = \big[\gamma_D\big(-\widetilde{\Delta}_{\Theta,\Omega} - z\widetilde{I}_\Omega\big)^{-1}\big]^* f \ \text{ in } L^2(\Omega; d^n x),\ f \in H^{-1/2}(\partial\Omega). \tag{4.26}$$

*As a consequence,*

$$\big[\gamma_D(-\Delta_{\Theta,\Omega} - zI_\Omega)^{-1}\big]^* f = \big[\gamma_D\big(-\widetilde{\Delta}_{\Theta,\Omega} - z\widetilde{I}_\Omega\big)^{-1}\big]^* f \ \text{ in } L^2(\Omega; d^n x),\ f \in L^2(\partial\Omega; d^{n-1}\omega). \tag{4.27}$$

We will need a similar compatibility result for the composition between the Neumann trace and resolvents of the Dirichlet Laplacian. To state it, we recall the restriction operator $R_\Omega$ in (2.18). Also, we denote by $I_{\mathbb{R}^n}$ the identity operator (for spaces of functions defined in $\mathbb{R}^n$). Finally, we recall the space (2.19) and the ultra weak Neumann trace operator $\widetilde{\gamma}_{\mathcal{N}}$ in (2.20), (2.21).

**Lemma 4.11.** *Assume Hypothesis* 2.1. *Then*

$$\big(\big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big)^{-1} \circ R_\Omega, I_{\mathbb{R}^n}\big) : (H^1(\Omega))^* \to W_z(\Omega), \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega}), \tag{4.28}$$

*is a well-defined, linear and bounded operator. Consequently,*

$$\widetilde{\gamma}_{\mathcal{N}}\big(\big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big)^{-1} \circ R_\Omega, I_{\mathbb{R}^n}\big) \in \mathcal{B}\big((H^1(\Omega))^*, H^{-1/2}(\partial\Omega)\big), \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega}), \tag{4.29}$$

*and, hence,*

$$\big[\widetilde{\gamma}_{\mathcal{N}}\big(\big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big)^{-1} \circ R_\Omega, I_{\mathbb{R}^n}\big)\big]^* \in \mathcal{B}\big(H^{1/2}(\partial\Omega), H^1(\Omega)\big), \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega}). \tag{4.30}$$

*Furthermore, the operators* (4.29), (4.30) *are compatible with* (3.13) *and* (3.14), *respectively, in the sense that for each* $z \in \mathbb{C}\backslash\sigma(-\Delta_{D,\Omega})$,

$$\widetilde{\gamma}_N(-\Delta_{D,\Omega} - zI_\Omega)^{-1} f = \widetilde{\gamma}_{\mathcal{N}}\big(\big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big)^{-1} \circ R_\Omega, I_{\mathbb{R}^n}\big) f \ \text{ in } H^{-1/2}(\partial\Omega),\ f \in L^2(\Omega; d^n x), \tag{4.31}$$

*and*

$$\begin{split}\big[\widetilde{\gamma}_N(-\Delta_{D,\Omega} - zI_\Omega)^{-1}\big]^* f = \big[\widetilde{\gamma}_{\mathcal{N}}\big(\big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big)^{-1} \circ R_\Omega, I_{\mathbb{R}^n}\big)\big]^* f \ \text{ in } L^2(\Omega; d^n x),\\ \text{for every element } f \in H^{1/2}(\partial\Omega).\end{split} \tag{4.32}$$

This yields the following $L^2$-version of Lemma 4.8.

**Lemma 4.12.** *Assume Hypothesis* 3.1 *and suppose that* $z \in \mathbb{C}\backslash(\sigma(-\Delta_{\Theta,\Omega}) \cup \sigma(-\Delta_{D,\Omega}))$. *Then the following resolvent relation holds on* $L^2(\Omega; d^n x)$,

$$\begin{aligned}(-\Delta_{\Theta,\Omega} - zI_\Omega)^{-1} &= (-\Delta_{D,\Omega} - zI_\Omega)^{-1} + \big[\gamma_D(-\Delta_{\Theta,\Omega} - \overline{z}I_\Omega)^{-1}\big]^*\big[\widetilde{\gamma}_N(-\Delta_{D,\Omega} - zI_\Omega)^{-1}\big] \\ &= (-\Delta_{D,\Omega} - zI_\Omega)^{-1} + \big[\widetilde{\gamma}_{\mathcal{N}}(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}\big]^*\big[\gamma_D(-\Delta_{\Theta,\Omega} - zI_\Omega)^{-1}\big].\end{aligned} \tag{4.33}$$

We note that the special case $\Theta = 0$ in Lemma 4.12 was discussed by Nakamura [61] (in connection with cubic boxes $\Omega$) and subsequently in [32, Lemma A.3] (in the case of a Lipschitz domain with a compact boundary).

We also recall the following useful result.

**Lemma 4.13.** *Assume Hypothesis* 4.1 *and suppose that* $z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta,\Omega})$. *Then*

$$\big[\widetilde{M}^{(0)}_{\Theta,D,\Omega}(z)\big]^* = \widetilde{M}^{(0)}_{\Theta,D,\Omega}(\overline{z}) \tag{4.34}$$

*as operators in* $\mathcal{B}\big(H^{-1/2}(\partial\Omega); H^{1/2}(\partial\Omega)\big)$. *In particular, assuming Hypothesis* 3.1, *then*

$$\big[M^{(0)}_{\Theta,D,\Omega}(z)\big]^* = M^{(0)}_{\Theta,D,\Omega}(\overline{z}). \tag{4.35}$$

Next we briefly recall the Herglotz property of the Robin-to-Dirichlet map. We recall that an operator-valued function $M(z) \in \mathcal{B}(\mathcal{H})$, $z \in \mathbb{C}_+$ (where $\mathbb{C}_+ = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0)$, for some separable complex Hilbert space $\mathcal{H}$, is called an *operator-valued Herglotz function* if $M(\,\cdot\,)$ is analytic on $\mathbb{C}_+$ and

$$\mathrm{Im}(M(z)) \geqslant 0, \quad z \in \mathbb{C}_+. \tag{4.36}$$

Here, as usual, $\mathrm{Im}(M) = (M - M^*)/(2i)$.

**Lemma 4.14.** *Assume Hypothesis* 4.1 *and suppose that* $z \in \mathbb{C}_+$. *Then for every* $g \in H^{-1/2}(\partial\Omega)$, $g \neq 0$, *one has*

$$\frac{1}{2i}\big\langle\, g, \big[\widetilde{M}_{\Theta,D}(z) - \widetilde{M}_{\Theta,D}(z)^*\big]g\big\rangle_{1/2} = \mathrm{Im}(z)\|u_\Theta\|^2_{L^2(\Omega;d^n x)} > 0, \tag{4.37}$$

*where* $u_\Theta$ *satisfies*

$$\begin{cases}(-\Delta - z)u = 0 \text{ in } \Omega, \quad u \in H^1(\Omega), \\ \big(\widetilde{\gamma}_N + \widetilde{\Theta}\gamma_D\big)u = g \text{ on } \partial\Omega.\end{cases} \tag{4.38}$$

*In particular, assuming Hypothesis* 3.1, *then*

$$\mathrm{Im}\big(M^{(0)}_{\Theta,D,\Omega}(z)\big) \geqslant 0, \quad z \in \mathbb{C}_+, \tag{4.39}$$

*and hence* $M^{(0)}_{\Theta,D,\Omega}(\,\cdot\,)$ *is an operator-valued Herglotz function on* $L^2(\partial\Omega; d^{n-1}\omega)$.

The following result represents a first variant of Krein's resolvent formula relating $\widetilde{\Delta}_{\Theta,\Omega}$ and $\widetilde{\Delta}_{D,\Omega}$ recently proved in [35]:

**Theorem 4.15.** *Assume Hypothesis* 4.1 *and suppose that* $z \in \mathbb{C}\backslash(\sigma(-\Delta_{\Theta,\Omega}) \cup \sigma(-\Delta_{D,\Omega}))$. *Then the following Krein formula holds on* $\big(H^1(\Omega)\big)^*$,

$$\begin{aligned}&\big(-\widetilde{\Delta}_{\Theta,\Omega} - z\widetilde{I}_\Omega\big)^{-1} = \big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big)^{-1} \circ R_\Omega \\ &\quad + \big[\widetilde{\gamma}_{\mathcal{N}}\big(\big(-\widetilde{\Delta}_{D,\Omega} - \overline{z}\widetilde{I}_\Omega\big)^{-1} \circ R_\Omega, I_{\mathbb{R}^n}\big)\big]^* \widetilde{M}^{(0)}_{\Theta,D,\Omega}(z)\big[\widetilde{\gamma}_{\mathcal{N}}\big(\big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big)^{-1} \circ R_\Omega, I_{\mathbb{R}^n}\big)\big].\end{aligned} \tag{4.40}$$

The following result details the $L^2(\Omega; d^n x)$-variant of Krein's formula:

**Theorem 4.16.** *Assume Hypothesis* 3.1 *and suppose that* $z \in \mathbb{C}\backslash(\sigma(-\Delta_{\Theta,\Omega}) \cup \sigma(-\Delta_{D,\Omega}))$. *Then the following Krein formula holds on* $L^2(\Omega; d^n x)$*:*

$$\begin{aligned}(-\Delta_{\Theta,\Omega} - zI_\Omega)^{-1} &= (-\Delta_{D,\Omega} - zI_\Omega)^{-1} \\ &\quad + \big[\widetilde{\gamma}_N(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* M^{(0)}_{\Theta,D,\Omega}(z)\big[\widetilde{\gamma}_N(-\Delta_{D,\Omega} - zI_\Omega)^{-1}\big].\end{aligned} \tag{4.41}$$

It should be noted that, by Lemma 3.4, the composition of operators in the right-hand side of (4.41) acts in a well-defined manner on $L^2(\Omega; d^n x)$.

An attractive feature of the Krein-type formula (4.41) lies in the fact that $M^{(0)}_{\Theta,D,\Omega}(z)$ encodes spectral information about $\Delta_{\Theta,\Omega}$. This will be pursued in future work.

Assuming Hypothesis 2.1, the special case $\Theta = 0$ then connects the Neumann and Dirichlet resolvents,

$$\begin{aligned}&\big(-\widetilde{\Delta}_{N,\Omega} - z\widetilde{I}_\Omega\big)^{-1} = \big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big)^{-1} \circ R_\Omega \\ &\quad + \big[\widetilde{\gamma}_{\mathcal{N}}\big(\big(-\widetilde{\Delta}_{D,\Omega} - \overline{z}\widetilde{I}_\Omega\big)^{-1} \circ R_\Omega, I_{\mathbb{R}^n}\big)\big]^* \widetilde{M}^{(0)}_{N,D,\Omega}(z)\big[\widetilde{\gamma}_{\mathcal{N}}\big(\big(-\widetilde{\Delta}_{D,\Omega} - z\widetilde{I}_\Omega\big)^{-1} \circ R_\Omega, I_{\mathbb{R}^n}\big)\big], \\ &\qquad\qquad z \in \mathbb{C}\backslash(\sigma(-\Delta_{N,\Omega}) \cup \sigma(-\Delta_{D,\Omega})),\end{aligned} \tag{4.42}$$

on $\big(H^1(\Omega)\big)^*$, and similarly,

$$\begin{aligned}(-\Delta_{N,\Omega} - zI_\Omega)^{-1} &= (-\Delta_{D,\Omega} - zI_\Omega)^{-1} \\ &\quad + \big[\widetilde{\gamma}_N(-\Delta_{D,\Omega} - \overline{z}I_\Omega)^{-1}\big]^* M^{(0)}_{N,D,\Omega}(z)\big[\widetilde{\gamma}_N(-\Delta_{D,\Omega} - zI_\Omega)^{-1}\big], \\ &\qquad\qquad z \in \mathbb{C}\backslash(\sigma(-\Delta_{N,\Omega}) \cup \sigma(-\Delta_{D,\Omega})),\end{aligned} \tag{4.43}$$

on $L^2(\Omega; d^n x)$. Here $\widetilde{M}^{(0)}_{N,D,\Omega}(z)$ and $M^{(0)}_{N,D,\Omega}(z)$ denote the corresponding Neumann-to-Dirichlet operators.

Due to the fundamental importance of Krein-type resolvent formulas (and more generally, Robin-to-Dirichlet maps) in connection with the spectral and inverse spectral theory of ordinary and partial differential operators, abstract versions, connected to boundary value spaces (boundary triples) and self-adjoint extensions of closed symmetric operators with equal (possibly infinite) deficiency spaces, have received enormous attention in the literature. In particular, we note that Robin-to-Dirichlet maps in the context of ordinary differential operators reduce to the celebrated (possibly, matrix-valued) Weyl–Titchmarsh function, the basic object of spectral analysis in this context. Since it is impossible to cover the literature in this paper, we refer, for instance, to [1, Sect. 84], [3], [7], [8], [11], [13], [14], [18], [20], [21], [33], [34], [39, Ch. 13], [41], [43]–[50], [53], [54], [59], [62]–[69], [72], [75]–[77], and the references cited therein. We add, however, that the case of infinite deficiency indices in the context of partial differential operators (in our concrete case, related to the deficiency indices of the operator closure of $-\Delta \restriction_{C_0^\infty(\Omega)}$ in $L^2(\Omega; d^n x)$), is much less studied and the results obtained in this section, especially, under the assumption of Lipschitz (i.e., minimally smooth) domains, to the best of our knowledge, are new.

Finally, we emphasize once more that Remark 3.7 also applies to the content of this section (assuming that $V$ is real-valued in connection with Lemmas 4.13 and 4.14).

## 5. Some Variants of Krein's Resolvent Formula Involving Robin-to-Robin Maps

In this section we present our principal results, variants of Krein's formula for the difference of resolvents of generalized Robin Laplacians corresponding to two different Robin boundary conditions on bounded Lipschitz domains. To the best of our knowledge, the results in this section are new.

**Hypothesis 5.1.** *Assume that the conditions in Hypothesis* 2.2 *are satisfied by two sesquilinear forms* $a_{\Theta_1}, a_{\Theta_2}$ *and, in addition,*

$$\widetilde{\Theta}_1, \widetilde{\Theta}_2 \in \mathcal{B}_\infty\big(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\big). \tag{5.1}$$

**Lemma 5.2.** *Assume Hypothesis* 5.1 *and suppose that* $z \in \mathbb{C}\backslash(\sigma(-\Delta_{\Theta_1,\Omega}) \cup \sigma(-\Delta_{\Theta_2,\Omega}))$. *Then the following resolvent relation holds on* $\big(H^1(\Omega)\big)^*$,

$$\begin{aligned} \big(-\widetilde{\Delta}_{\Theta_1,\Omega} - z\widetilde{I}_\Omega\big)^{-1} &= \big(-\widetilde{\Delta}_{\Theta_2,\Omega} - z\widetilde{I}_\Omega\big)^{-1} \\ &\quad + \big(-\widetilde{\Delta}_{\Theta_1,\Omega} - z\widetilde{I}_\Omega\big)^{-1}\gamma_D^*\big(\widetilde{\Theta}_1 - \widetilde{\Theta}_2\big)\gamma_D\big(-\widetilde{\Delta}_{\Theta_2,\Omega} - z\widetilde{I}_\Omega\big)^{-1}. \end{aligned} \tag{5.2}$$

*Proof.* To set the stage, we recall (2.14) and (2.15). Together with (4.12) and (4.16), these ensure that the composition of operators appearing on the right-hand side of (4.20) is well-defined. Next, let $\phi_1, \phi_2 \in L^2(\Omega; d^n x)$ be arbitrary and define

$$\begin{aligned} \psi_1 &= (-\Delta_{\Theta_1,\Omega} - \overline{z}I_\Omega)^{-1}\phi_1 \in \mathrm{dom}(\Delta_{\Theta_1,\Omega}) \subset H^1(\Omega), \\ \psi_2 &= (-\Delta_{\Theta_2,\Omega} - zI_\Omega)^{-1}\phi_2 \in \mathrm{dom}(\Delta_{\Theta_2,\Omega}) \subset H^1(\Omega). \end{aligned} \tag{5.3}$$

As a consequence of our earlier results, both sides of (4.20) are bounded operators from $(H^1(\Omega))^*$ into $H^1(\Omega)$. Since $L^2(\Omega; d^n x) \hookrightarrow \big(H^1(\Omega)\big)^*$ densely, it therefore suffices to show that the following identity holds:

$$\begin{aligned} &(\phi_1, (-\Delta_{\Theta_1,\Omega} - zI_\Omega)^{-1}\phi_2)_{L^2(\Omega;d^n x)} - (\phi_1, (-\Delta_{\Theta_2,\Omega} - zI_\Omega)^{-1}\phi_2)_{L^2(\Omega;d^n x)} \\ &\quad = \big(\phi_1, (-\Delta_{\Theta_1,\Omega} - zI_\Omega)^{-1}\gamma_D^*\big(\widetilde{\Theta}_1 - \widetilde{\Theta}_2\big)\gamma_D(-\Delta_{\Theta_2,\Omega} - zI_\Omega)^{-1}\phi_2\big)_{L^2(\Omega;d^n x)}. \end{aligned} \tag{5.4}$$

We note that according to (5.3) one has,

$$(\phi_1, (-\Delta_{\Theta_1,\Omega} - zI_\Omega)^{-1}\phi_2)_{L^2(\Omega;d^n x)} = ((-\Delta_{\Theta_1,\Omega} - \overline{z}I_\Omega)\psi_1, \psi_2)_{L^2(\Omega;d^n x)}, \tag{5.5}$$

$$\begin{aligned} (\phi_1, (-\Delta_{\Theta_2,\Omega} - zI_\Omega)^{-1}\phi_2)_{L^2(\Omega;d^n x)} &= \big(\big((-\Delta_{\Theta_2,\Omega} - zI_\Omega)^{-1}\big)^*\phi_1, \phi_2\big)_{L^2(\Omega;d^n x)} \\ &= ((-\Delta_{\Theta_2,\Omega} - \overline{z}I_\Omega)^{-1}\phi_1, \phi_2)_{L^2(\Omega;d^n x)} \\ &= (\psi_1, (-\Delta_{\Theta_2,\Omega} - zI_\Omega)\psi_2)_{L^2(\Omega;d^n x)}, \end{aligned} \tag{5.6}$$

and, further,

$$\begin{aligned} &\big(\phi_1, (-\Delta_{\Theta_1,\Omega} - zI_\Omega)^{-1}\gamma_D^*\big(\widetilde{\Theta}_1 - \widetilde{\Theta}_2\big)\gamma_D(-\Delta_{\Theta_2,\Omega} - zI_\Omega)^{-1}\phi_2\big)_{L^2(\Omega;d^n x)} \\ &\quad = {}_{H^1(\Omega)}\big\langle(-\Delta_{\Theta_1,\Omega} - \overline{z}I_\Omega)^{-1}\phi_1, \gamma_D^*\big(\widetilde{\Theta}_1 - \widetilde{\Theta}_2\big)\gamma_D(-\Delta_{\Theta_2,\Omega} - zI_\Omega)^{-1}\phi_2\big\rangle_{(H^1(\Omega))^*} \\ &\quad = \big\langle\gamma_D(-\Delta_{\Theta_1,\Omega} - \overline{z}I_\Omega)^{-1}\phi_1, \big(\widetilde{\Theta}_1 - \widetilde{\Theta}_2\big)\gamma_D(-\Delta_{\Theta_2,\Omega} - zI_\Omega)^{-1}\phi_2\big\rangle_{1/2} \\ &\quad = \big\langle\gamma_D\psi_1, \big(\widetilde{\Theta}_1 - \widetilde{\Theta}_2\big)\gamma_D\psi_2\big\rangle_{1/2}. \end{aligned} \tag{5.7}$$

Thus, matters have been reduced to proving that

$$\begin{aligned} &((-\Delta_{\Theta_1,\Omega} - \overline{z}I_\Omega)\psi_1, \psi_2)_{L^2(\Omega;d^n x)} - (\psi_1, (-\Delta_{\Theta_2,\Omega} - zI_\Omega)\psi_2)_{L^2(\Omega;d^n x)} \\ &\quad = \big\langle\gamma_D\psi_1, \big(\widetilde{\Theta}_1 - \widetilde{\Theta}_2\big)\gamma_D\psi_2\big\rangle_{1/2}. \end{aligned} \tag{5.8}$$

Using (A.11) for the left-hand side of (5.8) one obtains

$$\begin{aligned} &((-\Delta_{\Theta_1,\Omega} - \overline{z}I_\Omega)\psi_1, \psi_2)_{L^2(\Omega;d^n x)} - (\psi_1, (-\Delta_{\Theta_2,\Omega} - zI_\Omega)\psi_2)_{L^2(\Omega;d^n x)} \\ &\quad = -(\Delta\psi_1, \psi_2)_{L^2(\Omega;d^n x)} + (\psi_1, \Delta\psi_2)_{L^2(\Omega;d^n x)} \\ &\quad = (\nabla\psi_1, \nabla\psi_2)_{L^2(\Omega;d^n x)^n} - \langle\widetilde{\gamma}_N\psi_1, \gamma_D\psi_2\rangle_{1/2} - (\nabla\psi_1, \nabla\psi_2)_{L^2(\Omega;d^n x)^n} + \langle\gamma_D\psi_1, \widetilde{\gamma}_N\psi_2\rangle_{1/2} \\ &\quad = -\langle\widetilde{\gamma}_N\psi_1, \gamma_D\psi_2\rangle_{1/2} + \langle\gamma_D\psi_1, \widetilde{\gamma}_N\psi_2\rangle_{1/2}. \end{aligned} \tag{5.9}$$

Observing that $\widetilde{\gamma}_N\psi_j = -\widetilde{\Theta}_j\gamma_D\psi_j$ since $\psi_j \in \operatorname{dom}(\Delta_{\Theta_j,\Omega})$, $j = 1, 2$, one concludes (5.8). $\square$

Assuming Hypothesis 5.1 we now introduce the Robin-to-Robin map $\widetilde{M}^{(0)}_{\Theta_1,\Theta_2,\Omega}(z)$ as follows,

$$\widetilde{M}^{(0)}_{\Theta_1,\Theta_2,\Omega}(z)\colon \begin{cases} H^{-1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \\ \qquad\qquad f \mapsto -\big(\widetilde{\gamma}_N + \widetilde{\Theta}_2\gamma_D\big)u_{\Theta_1}, \end{cases} \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta_1,\Omega}), \tag{5.10}$$

where $u_{\Theta_1}$ is the unique solution of

$$(-\Delta - z)u = 0 \text{ in } \Omega, \quad u \in H^1(\Omega), \quad \big(\widetilde{\gamma}_N + \widetilde{\Theta}_1\gamma_D\big)u = f \text{ on } \partial\Omega. \tag{5.11}$$

**Theorem 5.3.** *Assume Hypothesis* 5.1. *Then*

$$\widetilde{M}^{(0)}_{\Theta_1,\Theta_2,\Omega}(z) \in \mathcal{B}\big(H^{-1/2}(\partial\Omega)\big), \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta_1,\Omega}), \tag{5.12}$$

*and*

$$\widetilde{M}^{(0)}_{\Theta_1,\Theta_2,\Omega}(z) = -I_{\partial\Omega} + \big(\widetilde{\Theta}_1 - \widetilde{\Theta}_2\big)\widetilde{M}^{(0)}_{\Theta_1,D,\Omega}(z), \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta_1,\Omega}). \tag{5.13}$$

*In particular,*

$$\widetilde{M}^{(0)}_{\Theta_1,\Theta_2,\Omega}(z)\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big) = -\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)+\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)\widetilde{M}^{(0)}_{\Theta_1,D,\Omega}(z)\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big), \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta_1,\Omega}), \tag{5.14}$$

*and*

$$\big[\widetilde{M}^{(0)}_{\Theta_1,\Theta_2,\Omega}(z)\big(\widetilde{\Theta}_1 - \widetilde{\Theta}_2\big)\big]^* = \widetilde{M}^{(0)}_{\Theta_1,\Theta_2,\Omega}(\overline{z})\big(\widetilde{\Theta}_1 - \widetilde{\Theta}_2\big), \quad z \in \mathbb{C}\backslash\sigma(-\Delta_{\Theta_1,\Omega}). \tag{5.15}$$

*Also, if $z \in \mathbb{C}\backslash(\sigma(-\Delta_{\Theta_1,\Omega}) \cup \sigma(-\Delta_{\Theta_2,\Omega}))$, then*

$$\widetilde{M}^{(0)}_{\Theta_1,\Theta_2,\Omega}(z) = \widetilde{M}^{(0)}_{\Theta_2,\Theta_1,\Omega}(z)^{-1}. \tag{5.16}$$

*Proof.* The membership in (5.12) is a consequence of (5.10) and Theorem 4.2. To see (5.13), assume that $f \in H^{-1/2}(\partial\Omega)$ and denote by $u_{\Theta_1} \in H^1(\Omega)$ the unique function satisfying $(-\Delta - z)u_{\Theta_1} = 0$ in $\Omega$ and $(\widetilde{\gamma}_N + \widetilde{\Theta}_1\gamma_D)u_{\Theta_1} = f$ on $\partial\Omega$. Then

$$\begin{aligned} \widetilde{M}^{(0)}_{\Theta_1,\Theta_2,\Omega}(z)f &= -\big(\widetilde{\gamma}_N + \widetilde{\Theta}_2\gamma_D\big)u_{\Theta_1} = -\big(\widetilde{\gamma}_N + \widetilde{\Theta}_1\gamma_D\big)u_{\Theta_1} + \big(\widetilde{\Theta}_1 - \widetilde{\Theta}_2\big)\gamma_D u_{\Theta_1} \\ &= -f + \big(\widetilde{\Theta}_1 - \widetilde{\Theta}_2\big)\widetilde{M}^{(0)}_{\Theta_1,D,\Omega}(z)f, \end{aligned} \tag{5.17}$$

proving (5.13). Going further, (5.14) is a direct consequence of (5.13), and (5.15) is clear from (5.14) and Lemma 4.13. Finally, as far as (5.16) is concerned, if $f \in H^{-1/2}(\partial\Omega)$ and $u_{\Theta_2} \in H^1(\Omega)$ is the unique function satisfying $(-\Delta - z)u_{\Theta_2} = 0$ in $\Omega$ and $\big(\widetilde{\gamma}_N + \widetilde{\Theta}_2\gamma_D\big)u_{\Theta_2} = f$ on $\partial\Omega$, then $\widetilde{M}^{(0)}_{\Theta_2,\Theta_1,\Omega}(z)f = -\big(\widetilde{\gamma}_N + \widetilde{\Theta}_1\gamma_D\big)u_{\Theta_2}$. As a consequence, if $u_{\Theta_1} \in H^1(\Omega)$ is the unique function satisfying $(-\Delta - z)u_{\Theta_2} = 0$ in $\Omega$ and $\big(\widetilde{\gamma}_N + \widetilde{\Theta}_1\gamma_D\big)u_{\Theta_1} = -\big(\widetilde{\gamma}_N + \widetilde{\Theta}_1\gamma_D\big)u_{\Theta_2}$ on $\partial\Omega$, it follows that $u_{\Theta_2} = -u_{\Theta_1}$ so that $\widetilde{M}^{(0)}_{\Theta_1,\Theta_2,\Omega}(z)\widetilde{M}^{(0)}_{\Theta_2,\Theta_1,\Omega}(z)f = -\big(\widetilde{\gamma}_N + \widetilde{\Theta}_2\gamma_D\big)u_{\Theta_1} = \big(\widetilde{\gamma}_N + \widetilde{\Theta}_2\gamma_D\big)u_{\Theta_2} = f$. In a similar fashion, $\widetilde{M}^{(0)}_{\Theta_2,\Theta_1,\Omega}(z)\widetilde{M}^{(0)}_{\Theta_1,\Theta_2,\Omega}(z)f = f$, so (5.16) is proved. $\square$

**Theorem 5.4.** *Assume Hypothesis* 5.1 *and suppose that $z \in \mathbb{C}\backslash(\sigma(-\Delta_{\Theta_1,\Omega}) \cup \sigma(-\Delta_{\Theta_2,\Omega}))$. Then the following Krein formula holds:*

$$\begin{aligned} &(-\widetilde{\Delta}_{\Theta_1,\Omega} - z\widetilde{I}_\Omega)^{-1} = (-\widetilde{\Delta}_{\Theta_2,\Omega} - z\widetilde{I}_\Omega)^{-1} \\ &\quad + \big[\gamma_D(-\widetilde{\Delta}_{\Theta_2,\Omega} - \overline{z}\widetilde{I}_\Omega)^{-1}\big]^*\big[\big(\widetilde{M}^{(0)}_{\Theta_1,\Theta_2,\Omega}(z) + I_{\partial\Omega}\big)\big(\widetilde{\Theta}_1 - \widetilde{\Theta}_2\big)\big]\big[\gamma_D\big(-\widetilde{\Delta}_{\Theta_2,\Omega} - z\widetilde{I}_\Omega\big)^{-1}\big], \end{aligned} \tag{5.18}$$

*as operators on $\big(H^1(\Omega)\big)^*$.*

*Proof.* We first claim that

$$\begin{aligned}&\big(\widetilde\Theta_1-\widetilde\Theta_2\big)\gamma_D\big(-\widetilde\Delta_{\Theta_1,\Omega}-z\widetilde I_\Omega\big)^{-1}\\&\quad=\big(\widetilde\Theta_1-\widetilde\Theta_2\big)\gamma_D\big(-\widetilde\Delta_{\Theta_1,\Omega}-z\widetilde I_\Omega\big)^{-1}\gamma_D^*\big(\widetilde\Theta_1-\widetilde\Theta_2\big)\gamma_D\big(-\widetilde\Delta_{\Theta_2,\Omega}-z\widetilde I_\Omega\big)^{-1},\end{aligned}\tag{5.19}$$

as operators in $\mathcal{B}\big((H^1(\Omega))^*,H^1(\Omega)\big)$. To see this, consider an arbitrary $w\in\big(H^1(\Omega)\big)^*$, then introduce

$$v=\gamma_D^*\big(\widetilde\Theta_1-\widetilde\Theta_2\big)\gamma_D\big(-\widetilde\Delta_{\Theta_2,\Omega}-z\widetilde I_\Omega\big)^{-1}w\in\big(H^1(\Omega)\big)^*,\tag{5.20}$$

and observe that, under the identification (2.17), (A.13) yields

$$\operatorname{supp}(v)\subseteq\partial\Omega.\tag{5.21}$$

As far as (5.19) is concerned, the goal is to show that

$$\big(\widetilde\Theta_1-\widetilde\Theta_2\big)\gamma_D\big(-\widetilde\Delta_{\Theta_1,\Omega}-z\widetilde I_\Omega\big)^{-1}w=\big(\widetilde\Theta_1-\widetilde\Theta_2\big)\gamma_D\big(-\widetilde\Delta_{\Theta_1,\Omega}-z\widetilde I_\Omega\big)^{-1}v.\tag{5.22}$$

To this end, we observe from (5.2) that

$$\big(-\widetilde\Delta_{\Theta_1,\Omega}-z\widetilde I_\Omega\big)^{-1}w=\big(-\widetilde\Delta_{\Theta_2,\Omega}-z\widetilde I_\Omega\big)^{-1}w+\big(-\widetilde\Delta_{\Theta_1,\Omega}-z\widetilde I_\Omega\big)^{-1}v.\tag{5.23}$$

Hence, by linearity,

$$\begin{aligned}\widetilde\gamma_{\mathcal N}\big(\big(-\widetilde\Delta_{\Theta_1,\Omega}-z\widetilde I_\Omega\big)^{-1}w,w\big)&=\widetilde\gamma_{\mathcal N}\big(\big(-\widetilde\Delta_{\Theta_2,\Omega}-z\widetilde I_\Omega\big)^{-1}w,w\big)\\&\quad+\widetilde\gamma_{\mathcal N}\big(\big(-\widetilde\Delta_{\Theta_1,\Omega}-z\widetilde I_\Omega\big)^{-1}v,0\big).\end{aligned}\tag{5.24}$$

A word of explanation is in order here: First, by Remark 4.5, $\big(\big(-\widetilde\Delta_{\Theta_j,\Omega}-z\widetilde I_\Omega\big)^{-1}w,w\big)\in W_z(\Omega)$ for $j=1,2$, so the terms in the first line of (5.24) are well-defined in $H^{-1/2}(\partial\Omega)$ (cf. (2.20)). Second, thanks to (5.21), we have that $\big(\big(-\widetilde\Delta_{\Theta_1,\Omega}-z\widetilde I_\Omega\big)^{-1}v,0\big)\in W_z(\Omega)$, so the last term in (5.24) is also well-defined in $H^{-1/2}(\partial\Omega)$. Next, from the fact that the functions $\big(-\widetilde\Delta_{\Theta_j,\Omega}-z\widetilde I_\Omega\big)^{-1}w$, $j=1,2$, satisfy homogeneous Robin boundary conditions, one infers

$$\widetilde\gamma_{\mathcal N}\big(\big(-\widetilde\Delta_{\Theta_j,\Omega}-z\widetilde I_\Omega\big)^{-1}w,w\big)=-\widetilde\Theta_j\gamma_D\big(-\widetilde\Delta_{\Theta_j,\Omega}-z\widetilde I_\Omega\big)^{-1}w,\quad j=1,2.\tag{5.25}$$

In a similar fashion,

$$\begin{aligned}\widetilde\gamma_{\mathcal N}\big(\big(-\widetilde\Delta_{\Theta_1,\Omega}-z\widetilde I_\Omega\big)^{-1}v,0\big)&=\widetilde\gamma_{\mathcal N}\big(\big(-\widetilde\Delta_{\Theta_1,\Omega}-z\widetilde I_\Omega\big)^{-1}v,v\big)-\widetilde\gamma_{\mathcal N}\big(0,v\big)\\&=-\widetilde\Theta_1\gamma_D\big(-\widetilde\Delta_{\Theta_1,\Omega}-z\widetilde I_\Omega\big)^{-1}v-\widetilde\gamma_{\mathcal N}\big(0,v\big).\end{aligned}\tag{5.26}$$

To compute $\widetilde\gamma_{\mathcal N}\big(0,v\big)$, pick an arbitrary $\phi\in H^{1/2}(\partial\Omega)$ and assume that $\Phi\in H^1(\Omega)$ is such that $\gamma_D\Phi=\phi$. Then, based on (2.21) and (5.20), one has

$$\begin{aligned}\langle\phi,\widetilde\gamma_{\mathcal N}\big(0,v\big)\rangle_{1/2}&=-{}_{H^1(\Omega)}\langle\Phi,v\rangle_{(H^1(\Omega))^*}\\&=-{}_{H^1(\Omega)}\big\langle\Phi,\gamma_D^*\big(\widetilde\Theta_1-\widetilde\Theta_2\big)\gamma_D\big(-\widetilde\Delta_{\Theta_2,\Omega}-z\widetilde I_\Omega\big)^{-1}w\big\rangle_{(H^1(\Omega))^*}\\&=-\big\langle\gamma_D\Phi,\big(\widetilde\Theta_1-\widetilde\Theta_2\big)\gamma_D\big(-\widetilde\Delta_{\Theta_2,\Omega}-z\widetilde I_\Omega\big)^{-1}w\big\rangle_{1/2}\\&=-\big\langle\phi,\big(\widetilde\Theta_1-\widetilde\Theta_2\big)\gamma_D\big(-\widetilde\Delta_{\Theta_2,\Omega}-z\widetilde I_\Omega\big)^{-1}w\big\rangle_{1/2}.\end{aligned}\tag{5.27}$$

This shows that

$$\widetilde\gamma_{\mathcal N}\big(0,v\big)=-\big(\widetilde\Theta_1-\widetilde\Theta_2\big)\gamma_D\big(-\widetilde\Delta_{\Theta_2,\Omega}-z\widetilde I_\Omega\big)^{-1}w.\tag{5.28}$$

By plugging (5.25), (5.26), and (5.28) back into (5.24), one then arrives at

$$\begin{aligned}-\widetilde\Theta_1\gamma_D\big(-\widetilde\Delta_{\Theta_1,\Omega}-z\widetilde I_\Omega\big)^{-1}w&=-\widetilde\Theta_2\gamma_D\big(-\widetilde\Delta_{\Theta_2,\Omega}-z\widetilde I_\Omega\big)^{-1}w-\widetilde\Theta_1\gamma_D\big(-\widetilde\Delta_{\Theta_1,\Omega}-z\widetilde I_\Omega\big)^{-1}v\\&\quad+\big(\widetilde\Theta_1-\widetilde\Theta_2\big)\gamma_D\big(-\widetilde\Delta_{\Theta_2,\Omega}-z\widetilde I_\Omega\big)^{-1}w.\end{aligned}\tag{5.29}$$

Upon recalling from (5.23) that

$$\big(-\widetilde{\Delta}_{\Theta_2,\Omega}-z\widetilde{I}_\Omega\big)^{-1}w=\big(-\widetilde{\Delta}_{\Theta_1,\Omega}-z\widetilde{I}_\Omega\big)^{-1}w-\big(-\widetilde{\Delta}_{\Theta_1,\Omega}-z\widetilde{I}_\Omega\big)^{-1}v, \tag{5.30}$$

now (5.22) readily follows from (5.29), (5.30) and some simple algebra. This finishes the proof of (5.30).

Next, since (see (4.19))

$$\widetilde{M}^{(0)}_{\Theta_1,D,\Omega}(z)=\gamma_D(-\Delta_{\Theta_1,\Omega}-zI_\Omega)^{-1}\gamma_D^*,\quad z\in\mathbb{C}\backslash\sigma(-\Delta_{\Theta_1,\Omega}), \tag{5.31}$$

we may then transform (5.19) into

$$\begin{aligned}
&\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)\gamma_D\big(-\widetilde{\Delta}_{\Theta_1,\Omega}-zI_\Omega\big)^{-1}\\
&\quad=\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)\widetilde{M}^{(0)}_{\Theta_1,D,\Omega}(z)\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)\gamma_D\big(-\widetilde{\Delta}_{\Theta_2,\Omega}-zI_\Omega\big)^{-1}\\
&\quad=\big(\widetilde{M}^{(0)}_{\Theta_1,\Theta_2,\Omega}(z)+I_{\partial\Omega}\big)\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)\gamma_D\big(-\widetilde{\Delta}_{\Theta_2,\Omega}-zI_\Omega\big)^{-1},
\end{aligned} \tag{5.32}$$

where the last line is based on (5.13). Taking adjoints in (5.32) (written with $\overline{z}$ in place of $z$) then leads to

$$\begin{aligned}
&\big(-\widetilde{\Delta}_{\Theta_1,\Omega}-zI_\Omega\big)^{-1}\gamma_D^*\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)\\
&\quad=\big[\gamma_D\big(-\widetilde{\Delta}_{\Theta_2,\Omega}-\overline{z}I_\Omega\big)^{-1}\big]^*\big[\big(\widetilde{M}^{(0)}_{\Theta_1,\Theta_2,\Omega}(\overline{z})+I_{\partial\Omega}\big)\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)\big]^*\\
&\quad=\big[\gamma_D\big(-\widetilde{\Delta}_{\Theta_2,\Omega}-\overline{z}I_\Omega\big)^{-1}\big]^*\big(\widetilde{M}^{(0)}_{\Theta_1,\Theta_2,\Omega}(z)+I_{\partial\Omega}\big)\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big),
\end{aligned} \tag{5.33}$$

by (5.15). Replacing this back in (5.2) then readily yields (5.18). □

We are interested in proving an $L^2$-version of Krein's formula in Theorem 5.4. This requires the following strengthening of Hypothesis 5.1.

**Hypothesis 5.5.** *Assume that the conditions in Hypothesis* 2.2 *are satisfied by two sesquilinear forms $a_{\Theta_1}, a_{\Theta_2}$ and suppose in addition that,*

$$\widetilde{\Theta}_1,\widetilde{\Theta}_2\in\mathcal{B}_\infty\big(H^1(\partial\Omega),L^2(\partial\Omega;d^{n-1}\omega)\big). \tag{5.34}$$

We recall (cf. (4.3))) that Hypothesis 5.5 is indeed stronger than Hypothesis 5.1.

As a preliminary matter, we first discuss the $L^2$-version of Theorem 5.3.

**Theorem 5.6.** *Assume Hypothesis* 5.5. *Then the Robin-to-Robin map, originally consider as an operator $\widetilde{M}^{(0)}_{\Theta_1,\Theta_2,\Omega}(z)\in\mathcal{B}\big(H^{-1/2}(\partial\Omega)\big)$, $z\in\mathbb{C}\backslash\sigma(-\Delta_{\Theta_1,\Omega})$, extends* (*in a compatible fashion*) *to an operator*

$$M^{(0)}_{\Theta_1,\Theta_2,\Omega}(z)\in\mathcal{B}\big(L^2(\partial\Omega;d^{n-1}\omega)\big),\quad z\in\mathbb{C}\backslash\sigma(-\Delta_{\Theta_1,\Omega}), \tag{5.35}$$

*which, for every $z\in\mathbb{C}\backslash\sigma(-\Delta_{\Theta_1,\Omega})$, satisfies*

$$M^{(0)}_{\Theta_1,\Theta_2,\Omega}(z)=-I_{\partial\Omega}+\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)M^{(0)}_{\Theta_1,D,\Omega}(z), \tag{5.36}$$

$$M^{(0)}_{\Theta_1,\Theta_2,\Omega}(z)\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)=-\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)+\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)M^{(0)}_{\Theta_1,D,\Omega}(z)\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big), \tag{5.37}$$

$$\big[M^{(0)}_{\Theta_1,\Theta_2,\Omega}(z)\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)\big]^*=M^{(0)}_{\Theta_1,\Theta_2,\Omega}(\overline{z})\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big). \tag{5.38}$$

*Furthermore, if $z\in\mathbb{C}\backslash(\sigma(-\Delta_{\Theta_1,\Omega})\cup\sigma(-\Delta_{\Theta_2,\Omega}))$, then also*

$$M^{(0)}_{\Theta_1,\Theta_2,\Omega}(z)=M^{(0)}_{\Theta_2,\Theta_1,\Omega}(z)^{-1}. \tag{5.39}$$

*Proof.* That for each $z\in\mathbb{C}\backslash\sigma(-\Delta_{\Theta_1,\Omega})$ the mapping $\widetilde{M}^{(0)}_{\Theta_1,\Theta_2,\Omega}(z)\in\mathcal{B}\big(H^{-1/2}(\partial\Omega)\big)$ extends to an operator $M^{(0)}_{\Theta_1,\Theta_2,\Omega}(z)\in\mathcal{B}\big(L^2(\partial\Omega;d^{n-1}\omega)\big)$ is a consequence of Theorem 3.2 and (5.34). This justifies the claim about (5.35). Properties (5.36)–(5.39) then follow from (5.35), Theorem 5.3, and a density argument. $\square$

With these preparatory results in place we are ready to state and prove the following $L^2$-version of Krein's formula.

**Theorem 5.7.** *Assume Hypothesis* 5.5 *and suppose that* $z\in\mathbb{C}\backslash(\sigma(-\Delta_{\Theta_1,\Omega})\cup\sigma(-\Delta_{\Theta_2,\Omega}))$. *Then the following Krein formula holds on* $L^2(\Omega;d^nx)$*:*

$$\begin{split}&(-\Delta_{\Theta_1,\Omega}-zI_\Omega)^{-1}=(-\Delta_{\Theta_2,\Omega}-zI_\Omega)^{-1}\\&\quad+\big[\gamma_D(-\Delta_{\Theta_2,\Omega}-\overline{z}I_\Omega)^{-1}\big]^*\big[\big(M^{(0)}_{\Theta_1,\Theta_2,\Omega}(z)+I_{\partial\Omega}\big)\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)\big]\big[\gamma_D(-\Delta_{\Theta_2,\Omega}-zI_\Omega)^{-1}\big].\end{split}\tag{5.40}$$

*Proof.* We start by observing that the following operators are well-defined, linear and bounded:

$$(-\Delta_{\Theta_j,\Omega}-zI_\Omega)^{-1}\in\mathcal{B}\big(L^2(\Omega;d^nx)\big),\quad j=1,2,\tag{5.41}$$
$$\gamma_D(-\Delta_{\Theta_2,\Omega}-zI_\Omega)^{-1}\in\mathcal{B}\big(L^2(\Omega;d^nx),H^1(\partial\Omega)\big),\tag{5.42}$$
$$\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)\in\mathcal{B}\big(H^1(\partial\Omega),L^2(\partial\Omega;d^{n-1}\omega)\big),\tag{5.43}$$
$$\big(M^{(0)}_{\Theta_1,\Theta_2,\Omega}(z)+I_{\partial\Omega}\big)\in\mathcal{B}\big(L^2(\partial\Omega;d^{n-1}\omega)\big),\tag{5.44}$$
$$\big[\gamma_D(-\Delta_{\Theta_2,\Omega}-\overline{z}I_\Omega)^{-1}\big]^*\in\mathcal{B}\big(L^2(\partial\Omega;d^{n-1}\omega),L^2(\Omega;d^nx)\big)\big).\tag{5.45}$$

Indeed, (5.41) follows from the fact that $z\in\mathbb{C}\backslash\big(\sigma(-\Delta_{\Theta_1,\Omega})\cup\sigma(-\Delta_{\Theta_2,\Omega})\big)$, (5.42) is covered by (4.24), (5.43) is taken care of by (5.34), (5.44) follows from (5.35), and (5.45) is a consequence of (4.25). Altogether, this shows that both sides of (5.40) are bounded operators on $L^2(\Omega;d^nx)$. With this in hand, the desired conclusion follows from Theorem 5.4, (4.27) and the fact that the operators (4.11) and (4.12) are compatible. $\square$

We conclude by establishing the following Herglotz property for the Robin-to-Robin map composed (to the right) by $\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)$. Specifically we have the following result:

**Theorem 5.8.** *Assume Hypothesis* 5.1 *and suppose that* $z\in\mathbb{C}\backslash\sigma(-\Delta_{\Theta_1,\Omega})$. *Then the operator* $\widetilde{M}^{(0)}_{\Theta_1,\Theta_2}(z)\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)$, *and hence* $\big[\widetilde{M}^{(0)}_{\Theta_1,\Theta_2}(z)+I_\Omega\big]\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)$, *has the Herglotz property when considered as operators in* $\mathcal{B}\big(H^{-1/2}(\partial\Omega)\big)$.

*Consequently, if Hypothesis* 5.5 *is assumed and* $z\in\mathbb{C}\backslash\sigma(-\Delta_{\Theta_1,\Omega})$, *then* $M^{(0)}_{\Theta_1,\Theta_2}(z)\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)$ *and* $\big[M^{(0)}_{\Theta_1,\Theta_2}(z)+I_\Omega\big]\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)$ *also have the Herglotz property when considered as operators in* $\mathcal{B}\big(L^2(\partial\Omega;d^{n-1}\omega)\big)$.

*Proof.* By Theorem 5.6 it suffices to prove only the first part in the statement. To this end, we recall (5.13) in Theorem 5.3. Composing the latter on the right by $(\widetilde{\Theta}_1-\widetilde{\Theta}_2)$ then yields

$$\widetilde{M}^{(0)}_{\Theta_1,\Theta_2}(z)\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)=-(\widetilde{\Theta}_1-\widetilde{\Theta}_2)+\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)\widetilde{M}^{(0)}_{\Theta_1,D}(z)\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big),\quad z\in\mathbb{C}\backslash\sigma(-\Delta_{\Theta_1,\Omega}).\tag{5.46}$$

Consequently,

$$\begin{split}\operatorname{Im}\big[\widetilde{M}^{(0)}_{\Theta_1,\Theta_2}(z)\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)\big]&=\operatorname{Im}\big[\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)\widetilde{M}^{(0)}_{\Theta_1,D}(z)\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)\big]\\&=\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)\operatorname{Im}\big[\widetilde{M}^{(0)}_{\Theta_1,D}(z)\big]\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big),\quad z\in\mathbb{C}\backslash\sigma(-\Delta_{\Theta_1,\Omega}).\end{split}\tag{5.47}$$

Now one can use Lemma 4.14 in order to conclude that

$$\operatorname{Im}\big[\widetilde{M}^{(0)}_{\Theta_1,\Theta_2}(z)\big(\widetilde{\Theta}_1-\widetilde{\Theta}_2\big)\big]\geq 0,\tag{5.48}$$

as desired. $\square$

We note again that Remark 3.7 also applies to the content of this section (assuming that $V$ is real-valued in connection with (5.38) and Theorem 5.8).

## Appendix A. Properties of Sobolev Spaces and Boundary Traces for Lipschitz Domains

The purpose of this appendix is to recall some basic facts in connection with Sobolev spaces corresponding to Lipschitz domains $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, and their boundaries. For more details we refer again to [35].

In this manuscript we use the following notation for the standard Sobolev Hilbert spaces ($s \in \mathbb{R}$),

$$H^s(\mathbb{R}^n) = \bigg\{U \in \mathcal{S}(\mathbb{R}^n)' \,\bigg|\, \|U\|^2_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} d^n\xi \, \big|\widehat{U}(\xi)\big|^2 \big(1 + |\xi|^{2s}\big) < \infty\bigg\}, \tag{A.1}$$

$$H^s(\Omega) = \{u \in \mathcal{D}'(\Omega) \,|\, u = U|_\Omega \text{ for some } U \in H^s(\mathbb{R}^n)\}, \tag{A.2}$$

$$H^s_0(\Omega) = \{u \in H^s(\mathbb{R}^n) \,|\, \operatorname{supp}(u) \subseteq \overline{\Omega}\}. \tag{A.3}$$

Here $\mathcal{D}'(\Omega)$ denotes the usual set of distributions on $\Omega \subseteq \mathbb{R}^n$, $\Omega$ open and nonempty (with $\mathcal{D}(\Omega)$ standing for the space of test functions in $\Omega$), $\mathcal{S}(\mathbb{R}^n)'$ is the space of tempered distributions on $\mathbb{R}^n$, and $\widehat{U}$ denotes the Fourier transform of $U \in \mathcal{S}(\mathbb{R}^n)'$. It is then immediate that

$$H^{s_1}(\Omega) \hookrightarrow H^{s_0}(\Omega) \text{ for } -\infty < s_0 \leq s_1 < +\infty, \tag{A.4}$$

continuously and densely.

Next, we recall the definition of a Lipschitz-domain $\Omega \subseteq \mathbb{R}^n$, $\Omega$ open and nonempty, for convenience of the reader: Let $\mathcal{N}$ be a space of real-valued functions in $\mathbb{R}^{n-1}$. One calls a bounded domain $\Omega \subset \mathbb{R}^n$ of class $\mathcal{N}$ if there exists a finite open covering $\{\mathcal{O}_j\}_{1\leq j\leq N}$ of the boundary $\partial\Omega$ of $\Omega$ with the property that, for every $j \in \{1, ..., N\}$, $\mathcal{O}_j \cap \Omega$ coincides with the portion of $\mathcal{O}_j$ lying in the over-graph of a function $\varphi_j \in \mathcal{N}$ (considered in a new system of coordinates obtained from the original one via a rigid motion). If $\mathcal{N}$ is $\mathrm{Lip}\,(\mathbb{R}^{n-1})$, the space of real-valued functions satisfying a (global) Lipschitz condition in $\mathbb{R}^{n-1}$, is called a *Lipschitz domain*; cf. [74, p. 189], where such domains are called "minimally smooth". The classical theorem of Rademacher of almost everywhere differentiability of Lipschitz functions ensures that, for any Lipschitz domain $\Omega$, the surface measure $d^{n-1}\omega$ is well-defined on $\partial\Omega$ and that there exists an outward pointing normal vector $\nu$ at almost every point of $\partial\Omega$.

For a Lipschitz domain $\Omega \subset \mathbb{R}^n$ it is known that

$$\big(H^s(\Omega)\big)^* = H^{-s}(\Omega), \quad -1/2 < s < 1/2. \tag{A.5}$$

See [78] for this and other related properties. We also refer to our convention of using the *adjoint* (rather than the dual) space $X^*$ of a Banach space $X$ as described near the end of the introduction.

Next, assume that $\Omega \subset \mathbb{R}^n$ is the domain lying above the graph of a Lipschitz function $\varphi \colon \mathbb{R}^{n-1} \to \mathbb{R}$. Then for $0 \leq s \leqslant 1$, the Sobolev space $H^s(\partial\Omega)$ consists of functions $f \in L^2(\partial\Omega; d^{n-1}\omega)$ such that $f(x', \varphi(x'))$, as a function of $x' \in \mathbb{R}^{n-1}$, belongs to $H^s(\mathbb{R}^{n-1})$. In this scenario we set

$$H^s(\partial\Omega) = \big(H^{-s}(\partial\Omega)\big)^*, \quad -1 \leqslant s \leqslant 0. \tag{A.6}$$

To define $H^s(\partial\Omega)$, $0 \leq s \leqslant 1$, when $\Omega$ is a Lipschitz domain with compact boundary, we use a smooth partition of unity to reduce matters to the graph case. More precisely, if $0 \leq s \leq 1$ then $f \in H^s(\partial\Omega)$ if and only if the assignment $\mathbb{R}^{n-1} \ni x' \mapsto (\psi f)(x', \varphi(x'))$ is in $H^s(\mathbb{R}^{n-1})$ whenever $\psi \in C_0^\infty(\mathbb{R}^n)$ and $\varphi \colon \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function with the property that if $\Sigma$ is an appropriate rotation and translation of $\{(x', \varphi(x')) \in \mathbb{R}^n \,|\, x' \in \mathbb{R}^{n-1}\}$, then $(\operatorname{supp}(\psi) \cap \partial\Omega) \subset \Sigma$ (this appears to be folklore, but a proof will appear in [60, Proposition 2.4]). Then Sobolev spaces with a negative amount of smoothness are defined as in (A.6) above.

From the above characterization of $H^s(\partial\Omega)$ it follows that any property of Sobolev spaces (of order $s \in [-1,1]$) defined in Euclidean domains, which are invariant under multiplication by smooth, compactly supported functions as well as composition by bi-Lipschitz diffeomorphisms, readily extends to the setting of $H^s(\partial\Omega)$ (via localization and pull-back). As a concrete example, for each Lipschitz domain $\Omega$ with compact boundary, one has

$$H^s(\partial\Omega) \hookrightarrow H^{s-\varepsilon}(\partial\Omega) \text{ compactly if } 0 < \varepsilon \leq s \leq 1. \tag{A.7}$$

For additional background information in this context we refer, for instance, to [9], [10], [29, Chs. V, VI], [38, Ch. 1], [58, Ch. 3], [82, Sect. I.4.2].

Moving on, we next consider the following bounded linear map

$$\begin{cases} \big\{(w,f) \in L^2(\Omega;d^nx)^n \times \big(H^1(\Omega)\big)^* \,\big|\, \operatorname{div}(w) = f|_\Omega\big\} \to H^{-1/2}(\partial\Omega) = \big(H^{1/2}(\partial\Omega)\big)^* \\ \hfill w \mapsto \nu\cdot(w,f) \end{cases} \tag{A.8}$$

by setting

$${}_{H^{1/2}(\partial\Omega)}\langle \phi, \nu\cdot(w,f)\rangle_{(H^{1/2}(\partial\Omega))^*} = \int_\Omega d^nx\, \overline{\nabla\Phi(x)}\cdot w(x) + {}_{H^1(\Omega)}\langle \Phi, f\rangle_{(H^1(\Omega))^*} \tag{A.9}$$

whenever $\phi \in H^{1/2}(\partial\Omega)$ and $\Phi \in H^1(\Omega)$ is such that $\gamma_D\Phi = \phi$. Here ${}_{H^1(\Omega)}\langle \Phi, f\rangle_{(H^1(\Omega))^*}$ in (A.9) is the natural pairing between functionals in $\big(H^1(\Omega)\big)^*$ and elements in $H^1(\Omega)$ (which, in turn, is compatible with the (bilinear) distributional pairing). It should be remarked that the above definition is independent of the particular extension $\Phi \in H^1(\Omega)$ of $\phi$.

Going further, one can introduce the ultra weak Neumann trace operator $\widetilde{\gamma}_{\mathcal N}$ as follows:

$$\widetilde{\gamma}_{\mathcal N}\colon \begin{cases} \big\{(u,f) \in H^1(\Omega) \times \big(H^1(\Omega)\big)^* \,\big|\, \Delta u = f|_\Omega\big\} \to H^{-1/2}(\partial\Omega) \\ \hfill u \mapsto \widetilde{\gamma}_{\mathcal N}(u,f) = \nu\cdot(\nabla u, f), \end{cases} \tag{A.10}$$

with the dot product understood in the sense of (A.8). We emphasize that the ultra weak Neumann trace operator $\widetilde{\gamma}_{\mathcal N}$ in (A.10) is a re-normalization of the operator $\gamma_N$ introduced in (2.12) relative to the extension of $\Delta u \in H^{-1}(\Omega)$ to an element $f$ of the space $\big(H^1(\Omega)\big)^* = \{g \in H^{-1}(\mathbb{R}^n) \,|\, \operatorname{supp}(g) \subseteq \overline{\Omega}\}$. For the relationship between the weak and ultra weak Neumann trace operators, see (2.22)–(2.24). In addition, one can show that the ultra weak Neumann trace operator (A.10) is onto (indeed, this is a corollary of Theorem 4.4). We note that (A.9) and (A.10) yield the following Green's formula

$$\langle \gamma_D\Phi, \widetilde{\gamma}_{\mathcal N}(u,f)\rangle_{1/2} = (\nabla\Phi, \nabla u)_{L^2(\Omega;d^nx)^n} + {}_{H^1(\Omega)}\langle \Phi, f\rangle_{(H^1(\Omega))^*}, \tag{A.11}$$

valid for any $u \in H^1(\Omega)$, $f \in \big(H^1(\Omega)\big)^*$ with $\Delta u = f|_\Omega$, and any $\Phi \in H^1(\Omega)$. The pairing on the left-hand side of (A.11) is between functionals in $\big(H^{1/2}(\partial\Omega)\big)^*$ and elements in $H^{1/2}(\partial\Omega)$, whereas the last pairing on the right-hand side in (A.11) is between functionals in $\big(H^1(\Omega)\big)^*$ and elements in $H^1(\Omega)$. For further use, we also note that the adjoint of (2.7) maps boundedly as follows

$$\gamma_D^*\colon \big(H^{s-1/2}(\partial\Omega)\big)^* \to \big(H^s(\Omega)\big)^*, \quad 1/2 < s < 3/2. \tag{A.12}$$

Identifying $\big(H^s(\Omega)\big)^*$ with $H_0^{-s}(\Omega) \hookrightarrow H^{-s}(\mathbb{R}^n)$ (cf. Proposition 2.9 in [40]), it follows that

$$\operatorname{ran}(\gamma_D^*) \subseteq \{u \in H^{-s}(\mathbb{R}^n) \,|\, \operatorname{supp}(u) \subseteq \partial\Omega\}, \quad 1/2 < s < 3/2. \tag{A.13}$$

**Remark A.1.** While it is tempting to view $\gamma_D$ as an unbounded but densely defined operator on $L^2(\Omega;d^nx)$ whose domain contains the space $C_0^\infty(\Omega)$, one should note that in this case its adjoint $\gamma_D^*$ is not densely defined: Indeed (cf. [32, Remark A.4]), $\operatorname{dom}(\gamma_D^*) = \{0\}$ and hence $\gamma_D$ is not a closable linear operator in $L^2(\Omega;d^nx)$.

We conclude this appendix by recalling the following result from [36].

**Lemma A.2** (cf. [36], Lemma A.6)**.** *Suppose $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is an open Lipschitz domain with a compact, nonempty boundary $\partial\Omega$. Then the Dirichlet trace operator $\gamma_D$ (originally considered as in* (2.7)*) satisfies* (2.9).

## Appendix B. Sesquilinear Forms and Associated Operators

In this appendix we describe a few basic facts on sesquilinear forms and linear operators associated with them. A slightly more expanded version of this material appeared in [35, Appendix B].

Let $\mathcal{H}$ be a complex separable Hilbert space with scalar product $(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$ (antilinear in the first and linear in the second argument), $\mathcal{V}$ a reflexive Banach space continuously and densely embedded into $\mathcal{H}$. Then also $\mathcal{H}$ embeds continuously and densely into $\mathcal{V}^*$.

$$\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*. \tag{B.1}$$

Here the continuous embedding $\mathcal{H} \hookrightarrow \mathcal{V}^*$ is accomplished via the identification

$$\mathcal{H} \ni u \mapsto (\,\cdot\,, u)_{\mathcal{H}} \in \mathcal{V}^*, \tag{B.2}$$

and we recall the convention in this manuscript (cf. the discussion at the end of the introduction) that if $X$ denotes a Banach space, $X^*$ denotes the *adjoint space* of continuous conjugate linear functionals on $X$, also known as the *conjugate dual* of $X$.

In particular, if the sesquilinear form

$$_{\mathcal{V}}\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{V}^*} \colon \mathcal{V} \times \mathcal{V}^* \to \mathbb{C} \tag{B.3}$$

denotes the duality pairing between $\mathcal{V}$ and $\mathcal{V}^*$, then

$$_{\mathcal{V}}\langle u, v\rangle_{\mathcal{V}^*} = (u, v)_{\mathcal{H}}, \quad u \in \mathcal{V},\ v \in \mathcal{H} \hookrightarrow \mathcal{V}^*, \tag{B.4}$$

that is, the $\mathcal{V}, \mathcal{V}^*$ pairing $_{\mathcal{V}}\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{V}^*}$ is compatible with the scalar product $(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$ in $\mathcal{H}$.

Let $T \in \mathcal{B}(\mathcal{V}, \mathcal{V}^*)$. Since $\mathcal{V}$ is reflexive, $(\mathcal{V}^*)^* = \mathcal{V}$, one has

$$T \colon \mathcal{V} \to \mathcal{V}^*, \quad T^* \colon \mathcal{V} \to \mathcal{V}^* \tag{B.5}$$

and

$$_{\mathcal{V}}\langle u, Tv\rangle_{\mathcal{V}^*} = {}_{\mathcal{V}^*}\langle T^*u, v\rangle_{(\mathcal{V}^*)^*} = {}_{\mathcal{V}^*}\langle T^*u, v\rangle_{\mathcal{V}} = \overline{_{\mathcal{V}}\langle v, T^*u\rangle_{\mathcal{V}^*}}. \tag{B.6}$$

*Self-adjointness* of $T$ is then defined by $T = T^*$, that is,

$$_{\mathcal{V}}\langle u, Tv\rangle_{\mathcal{V}^*} = {}_{\mathcal{V}^*}\langle Tu, v\rangle_{\mathcal{V}} = \overline{_{\mathcal{V}}\langle v, Tu\rangle_{\mathcal{V}^*}}, \quad u, v \in \mathcal{V}, \tag{B.7}$$

*nonnegativity* of $T$ is defined by

$$_{\mathcal{V}}\langle u, Tu\rangle_{\mathcal{V}^*} \geq 0, \quad u \in \mathcal{V}, \tag{B.8}$$

and *boundedness from below of $T$ by $c_T \in \mathbb{R}$* is defined by

$$_{\mathcal{V}}\langle u, Tu\rangle_{\mathcal{V}^*} \geq c_T \|u\|_{\mathcal{H}}^2, \quad u \in \mathcal{V}. \tag{B.9}$$

(By (B.4), this is equivalent to $_{\mathcal{V}}\langle u, Tu\rangle_{\mathcal{V}^*} \geq c_T\, {}_{\mathcal{V}}\langle u, u\rangle_{\mathcal{V}^*}$, $u \in \mathcal{V}$.)

Next, let the sesquilinear form $a(\,\cdot\,,\,\cdot\,) \colon \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ (antilinear in the first and linear in the second argument) be *$\mathcal{V}$-bounded*, that is, there exists a $c_a > 0$ such that

$$|a(u, v)| \leqslant c_a \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}, \quad u, v \in \mathcal{V}. \tag{B.10}$$

Then $\widetilde{A}$ defined by

$$\widetilde{A} \colon \begin{cases} \mathcal{V} \to \mathcal{V}^*, \\ v \mapsto \widetilde{A}v = a(\,\cdot\,, v), \end{cases} \tag{B.11}$$

satisfies

$$\widetilde{A} \in \mathcal{B}(\mathcal{V}, \mathcal{V}^*) \ \text{ and } \ _{\mathcal{V}}\big\langle u, \widetilde{A}v\big\rangle_{\mathcal{V}^*} = a(u, v), \quad u, v \in \mathcal{V}. \tag{B.12}$$

Assuming further that $a(\,\cdot\,,\,\cdot\,)$ is *symmetric*, that is,

$$a(u,v) = \overline{a(v,u)}, \quad u,v \in \mathcal{V}, \tag{B.13}$$

and that $a$ is $\mathcal{V}$*-coercive*, that is, there exists a constant $C_0 > 0$ such that

$$a(u,u) \geq C_0 \|u\|_{\mathcal{V}}^2, \quad u \in \mathcal{V}, \tag{B.14}$$

respectively, then,

$$\widetilde{A}\colon \mathcal{V} \to \mathcal{V}^* \text{ is bounded, self-adjoint, and boundedly invertible.} \tag{B.15}$$

Moreover, denoting by $A$ the part of $\widetilde{A}$ in $\mathcal{H}$ defined by

$$\mathrm{dom}(A) = \big\{u \in \mathcal{V} \,|\, \widetilde{A}u \in \mathcal{H}\big\} \subseteq \mathcal{H}, \quad A = \widetilde{A}\big|_{\mathrm{dom}(A)}\colon \mathrm{dom}(A) \to \mathcal{H}, \tag{B.16}$$

then $A$ is a (possibly unbounded) self-adjoint operator in $\mathcal{H}$ satisfying

$$A \geq C_0 I_{\mathcal{H}}, \tag{B.17}$$

$$\mathrm{dom}\big(A^{1/2}\big) = \mathcal{V}. \tag{B.18}$$

In particular,

$$A^{-1} \in \mathcal{B}(\mathcal{H}). \tag{B.19}$$

The facts (B.1)–(B.19) are a consequence of the Lax–Milgram theorem and the second representation theorem for symmetric sesquilinear forms. Details can be found, for instance, in [24, §VI.3, §VII.1], [29, Ch. IV], and [52].

Next, consider a symmetric form $b(\,\cdot\,,\,\cdot\,)\colon \mathcal{V}\times\mathcal{V} \to \mathbb{C}$ and assume that $b$ is *bounded from below by* $c_b \in \mathbb{R}$, that is,

$$b(u,u) \geq c_b \|u\|_{\mathcal{H}}^2, \quad u \in \mathcal{V}. \tag{B.20}$$

Introducing the scalar product $(\,\cdot\,,\,\cdot\,)_{\mathcal{V}(b)}\colon \mathcal{V}\times\mathcal{V} \to \mathbb{C}$ (with associated norm $\|\cdot\|_{\mathcal{V}(b)}$) by

$$(u,v)_{\mathcal{V}(b)} = b(u,v) + (1-c_b)(u,v)_{\mathcal{H}}, \quad u,v \in \mathcal{V}, \tag{B.21}$$

turns $\mathcal{V}$ into a pre-Hilbert space $(\mathcal{V}; (\,\cdot\,,\,\cdot\,)_{\mathcal{V}(b)})$, which we denote by $\mathcal{V}(b)$. The form $b$ is called *closed* if $\mathcal{V}(b)$ is actually complete, and hence a Hilbert space. The form $b$ is called *closable* if it has a closed extension. If $b$ is closed, then

$$|b(u,v) + (1-c_b)(u,v)_{\mathcal{H}}| \leqslant \|u\|_{\mathcal{V}(b)} \|v\|_{\mathcal{V}(b)}, \quad u,v \in \mathcal{V}, \tag{B.22}$$

and

$$|b(u,u) + (1-c_b)\|u\|_{\mathcal{H}}^2| = \|u\|_{\mathcal{V}(b)}^2, \quad u \in \mathcal{V}, \tag{B.23}$$

show that the form $b(\,\cdot\,,\,\cdot\,) + (1-c_b)(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$ is a symmetric, $\mathcal{V}$-bounded, and $\mathcal{V}$-coercive sesquilinear form. Hence, by (B.11) and (B.12), there exists a linear map

$$\widetilde{B}_{c_b}\colon \begin{cases} \mathcal{V}(b) \to \mathcal{V}(b)^*, \\ \quad v \mapsto \widetilde{B}_{c_b} v = b(\,\cdot\,,v) + (1-c_b)(\,\cdot\,,v)_{\mathcal{H}}, \end{cases} \tag{B.24}$$

with

$$\widetilde{B}_{c_b} \in \mathcal{B}(\mathcal{V}(b), \mathcal{V}(b)^*) \text{ and } {}_{\mathcal{V}(b)}\big\langle u, \widetilde{B}_{c_b} v\big\rangle_{\mathcal{V}(b)^*} = b(u,v) + (1-c_b)(u,v)_{\mathcal{H}}, \quad u,v \in \mathcal{V}. \tag{B.25}$$

Introducing the linear map

$$\widetilde{B} = \widetilde{B}_{c_b} + (c_b - 1)\widetilde{I}\colon \mathcal{V}(b) \to \mathcal{V}(b)^*, \tag{B.26}$$

where $\widetilde{I}\colon \mathcal{V}(b) \hookrightarrow \mathcal{V}(b)^*$ denotes the continuous inclusion (embedding) map of $\mathcal{V}(b)$ into $\mathcal{V}(b)^*$, one obtains a self-adjoint operator $B$ in $\mathcal{H}$ by restricting $\widetilde{B}$ to $\mathcal{H}$,

$$\mathrm{dom}(B) = \big\{u \in \mathcal{V} \,\big|\, \widetilde{B}u \in \mathcal{H}\big\} \subseteq \mathcal{H}, \quad B = \widetilde{B}\big|_{\mathrm{dom}(B)}\colon \mathrm{dom}(B) \to \mathcal{H}, \tag{B.27}$$

satisfying the following properties:

$$B \geq c_b I_{\mathcal{H}}, \tag{B.28}$$

$$\mathrm{dom}\big(|B|^{1/2}\big) = \mathrm{dom}\big((B - c_b I_{\mathcal{H}})^{1/2}\big) = \mathcal{V}, \tag{B.29}$$

$$\begin{aligned} b(u,v) &= \big(|B|^{1/2}u, U_B|B|^{1/2}v\big)_{\mathcal{H}} && \text{(B.30)}\\ &= \big((B - c_b I_{\mathcal{H}})^{1/2}u, (B - c_b I_{\mathcal{H}})^{1/2}v\big)_{\mathcal{H}} + c_b(u,v)_{\mathcal{H}} && \text{(B.31)}\\ &= {}_{\mathcal{V}(b)}\big\langle u, \widetilde{B}v\big\rangle_{\mathcal{V}(b)^*}, \quad u,v \in \mathcal{V}, && \text{(B.32)} \end{aligned}$$

$$b(u,v) = (u, Bv)_{\mathcal{H}}, \quad u \in \mathcal{V}, \ v \in \mathrm{dom}(B), \tag{B.33}$$

$$\begin{aligned} \mathrm{dom}(B) = \{v \in \mathcal{V} \,|\, &\text{there exists an } f_v \in \mathcal{H} \text{ such that} \\ & b(w,v) = (w, f_v)_{\mathcal{H}} \text{ for all } w \in \mathcal{V}\}, \end{aligned} \tag{B.34}$$

$$\begin{aligned} &Bu = f_u, \quad u \in \mathrm{dom}(B), \\ &\mathrm{dom}(B) \text{ is dense in } \mathcal{H} \text{ and in } \mathcal{V}(b). \end{aligned} \tag{B.35}$$

Properties (B.34) and (B.35) uniquely determine $B$. Here $U_B$ in (B.31) is the partial isometry in the polar decomposition of $B$, that is,

$$B = U_B|B|, \quad |B| = (B^*B)^{1/2}. \tag{B.36}$$

The operator $B$ is called *the operator associated with the form $b$*.

The facts (B.20)–(B.35) comprise the second representation theorem of sesquilinear forms (cf. [29, Sect. IV.2], [30, Sects. 1.2–1.5], and [42, Sect. VI.2.6]).

**Acknowledgments.** We wish to thank Gerd Grubb for questioning an inaccurate claim in an earlier version of the paper and Maxim Zinchenko for helpful discussions on this topic. Fritz Gesztesy would like to thank all organizers of the international conference on Modern Analysis and Applications (MAA 2007), and especially, Vadym Adamyan, for their kind invitation, the stimulating atmosphere during the meeting, and the hospitality extended to him during his stay in Odessa in April of 2007. He is also indebted to Vyacheslav Pivovarchik for numerous assistance before and during this conference.

## References


[1] N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space*, Dover, New York, 1993.

[2] S. Albeverio, J. F. Brasche, M. M. Malamud, and H. Neidhardt, *Inverse spectral theory for symmetric operators with several gaps: scalar-type Weyl functions*, J. Funct. Anal. **228**, 144–188 (2005).

[3] S. Albeverio, M. Dudkin, A. Konstantinov, and V. Koshmanenko, *On the point spectrum of $\mathcal{H}_{-2}$-singular perturbations*, Math. Nachr. **280**, 20–27 (2007).

[4] W. O. Amrein and D. B. Pearson, *M operators: a generalization of Weyl–Titchmarsh theory*, J. Comp. Appl. Math. **171**, 1–26 (2004).

[5] W. Arendt and M. Warma, *Dirichlet and Neumann boundary conditions: What is in between?*, J. Evolution Eq. **3**, 119–135 (2003).

[6] W. Arendt and M. Warma, *The Laplacian with Robin boundary conditions on arbitrary domains*, Potential Anal. **19**, 341–363 (2003).

[7] Yu. M. Arlinskii and E. R. Tsekanovskii, *Some remarks on singular perturbations of self-adjoint operators*, Meth. Funct. Anal. Top. **9**, No. 4, 287–308 (2003).

[8] Yu. Arlinskii and E. Tsekanovskii, *The von Neumann problem for nonnegative symmetric operators*, Int. Eq. Operator Theory, **51**, 319–356 (2005).

[9] G. Auchmuty, *Steklov eigenproblems and the representation of solutions of elliptic boundary value problems*, Num. Funct. Anal. Optimization **25**, 321–348 (2004).

[10] G. Auchmuty, *Spectral characterization of the trace spaces $H^s(\partial\Omega)$*, SIAM J. Math. Anal. **38**, 894–905 (2006).

[11] J. Behrndt and M. Langer, *Boundary value problems for partial differential operators on bounded domains*, J. Funct. Anal. **243**, 536–565 (2007).

[12] J. Behrndt, M. M. Malamud, and H. Neidhardt, *Scattering matrices and Weyl functions*, preprint, 2006.
[13] S. Belyi, G. Menon, and E. Tsekanovskii, *On Krein's formula in the case of non-densely defined symmetric operators*, J. Math. Anal. Appl. **264**, 598–616 (2001).
[14] S. Belyi and E. Tsekanovskii, *On Krein's formula in indefinite metric spaces*, Lin. Algebra Appl. **389**, 305–322 (2004).
[15] M. Biegert and M. Warma, *Removable singularities for a Sobolev space*, J. Math. Anal. Appl. **313**, 49–63 (2006).
[16] J. F. Brasche, M. M. Malamud, and H. Neidhardt, *Weyl functions and singular continuous spectra of self-adjoint extensions*, in *Stochastic Processes, Physics and Geometry: New Interplays. II. A Volume in Honor of Sergio Albeverio*, F. Gesztesy, H. Holden, J. Jost, S. Paycha, M. Röckner, and S. Scarlatti (eds.), Canadian Mathematical Society Conference Proceedings, Vol. 29, Amer. Math. Soc., Providence, RI, 2000, pp. 75–84.
[17] J. F. Brasche, M. M. Malamud, and H. Neidhardt, *Weyl function and spectral properties of self-adjoint extensions*, Integral Eqs. Operator Theory **43**, 264–289 (2002).
[18] B. M. Brown, G. Grubb, and I. G. Wood, *M-functions for closed extensions of adjoint pairs of operators with applications to elliptic boundary problems*, preprint, 2008, arXiv:0803.3630.
[19] B. M. Brown and M. Marletta, *Spectral inclusion and spectral exactness for PDE's on exterior domains*, IMA J. Numer. Anal. **24**, 21–43 (2004).
[20] B. M. Brown, M. Marletta, S. Naboko, and I. Wood, *Boundary triplets and M-functions for non-selfadjoint operators, with applications to elliptic PDEs and block operator matrices*, J. London Math. Soc., to appear.
[21] J. Brüning, V. Geyler, and K. Pankrashkin, *Spectra of self-adjoint extensions and applications to solvable Schrödinger operators*, preprint, 2007.
[22] E. N. Dancer and D. Daners, *Domain perturbation for elliptic equations subject to Robin boundary conditions*, J. Diff. Eq. **138**, 86–132 (1997).
[23] D. Daners, *Robin boundary value problems on arbitrary domains*, Trans. Amer. Math. Soc. **352**, 4207–4236 (2000).
[24] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology, Volume 2, Functional and Variational Methods*, Springer, Berlin, 2000.
[25] V. A. Derkach, S. Hassi, M. M. Malamud, and H. S. V. de Snoo, *Generalized resolvents of symmetric operators and admissibility*, Meth. Funct. Anal. Top. **6**, No. 3, 24–55 (2000).
[26] V. Derkach, S. Hassi, M. Malamud, and H. de Snoo, *Boundary relations and their Weyl families*, Trans. Amer. Math. Soc. **358**, 5351–5400 (2006).
[27] V. A. Derkach and M. M. Malamud, *Generalized resolvents and the boundary value problems for Hermitian operators with gaps*, J. Funct. Anal. **95**, 1–95 (1991).
[28] V. A. Derkach and M. M. Malamud, *The extension theory of Hermitian operators and the moment problem*, J. Math. Sci. **73**, 141–242 (1995).
[29] D. E. Edmunds and W. D. Evans, *Spectral Theory and Differential Operators*, Clarendon Press, Oxford, 1989.
[30] W. G. Faris, *Self-Adjoint Operators*, Lecture Notes in Mathematics, Vol. **433**, Springer, Berlin, 1975.
[31] F. Gesztesy, N. J. Kalton, K. A. Makarov, and E. Tsekanovskii, *Some Applications of Operator-Valued Herglotz Functions*, in *Operator Theory, System Theory and Related Topics. The Moshe Livšic Anniversary Volume*, D. Alpay and V. Vinnikov (eds.), Operator Theory: Advances and Applications, Vol. 123, Birkhäuser, Basel, 2001, p. 271–321.
[32] F. Gesztesy, Y. Latushkin, M. Mitrea, and M. Zinchenko, *Nonselfadjoint operators, infinite determinants, and some applications*, Russ. J. Math. Phys. **12**, 443–471 (2005).
[33] F. Gesztesy, K. A. Makarov, and E. Tsekanovskii, *An addendum to Krein's formula*, J. Math. Anal. Appl. **222**, 594–606 (1998).
[34] F. Gesztesy and M. M. Malamud, *Boundary value problems for elliptic operators from the extension theory point of view and Weyl–Titchmarsh functions*, in preparation.
[35] F. Gesztesy and M. Mitrea, *Generalized Robin Boundary Conditions, Robin-to-Dirichlet Maps, and Krein-Type Resolvent Formulas for Schrödinger Operators on Bounded Lipschitz Domains*, preprint, 2008, to appear in *Perspectives in Partial Differential Equations, Harmonic Analysis and Applications*, D. Mitrea and M. Mitrea (eds.), Proceedings of Symposia in Pure Mathematics, American Mathematical Society, Providence, RI, 2008.
[36] F. Gesztesy, M. Mitrea, and M. Zinchenko, *Variations on a Theme of Jost and Pais*, J. Funct. Anal. **253**, 399–448 (2007).
[37] V. I. Gorbachuk and M. L. Gorbachuk, *Boundary Value Problems for Operator Differential Equations*, Kluwer, Dordrecht, 1991.
[38] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston, 1985.
[39] G. Grubb, *Distributions and Operators*, Graduate Texts in Mathematics, Springer, in preparation.
[40] D. Jerison and C. Kenig, *The inhomogeneous Dirichlet problem in Lipschitz domains*, J. Funct. Anal. **130**, 161–219 (1995).

[41] W. Karwowski and S. Kondej, *The Laplace operator, null set perturbations and boundary conditions*, in *Operator Methods in Ordinary and Partial Differential Equations*, S. Albeverio, N. Elander, W. N. Everitt, and P. Kurasov (eds.), Operator Theory: Advances and Applications, Vol. 132, Birkhäuser, Basel, 2002, pp. 233–244.

[42] T. Kato, *Perturbation Theory for Linear Operators*, corr. printing of the 2nd ed., Springer, Berlin, 1980.

[43] V. Koshmanenko, *Singular operators as a parameter of self-adjoint extensions*, in *Operator Theory and Related Topics*, V. M. Adamyan, I. Gohberg, M. Gorbachuk, V. Gorbachuk, M. A. Kaashoek, H. Langer, and G. Popov (eds.), Operator Theory: Advances and Applications, Vol. 118, Birkhäuser, Basel, 2000, pp. 205–223.

[44] M. G. Krein, *On Hermitian operators with deficiency indices one*, Dokl. Akad. Nauk SSSR **43**, 339–342 (1944). (Russian).

[45] M. G. Krein, *Resolvents of a hermitian operator with defect index* $(m, m)$, Dokl. Akad. Nauk SSSR **52**, 657–660 (1946). (Russian).

[46] M. G. Krein and I. E. Ovcharenko, *Q-functions and sc-resolvents of nondensely defined hermitian contractions*, Sib. Math. J. **18**, 728–746 (1977).

[47] M. G. Krein and I. E. Ovčarenko, *Inverse problems for Q-functions and resolvent matrices of positive hermitian operators*, Sov. Math. Dokl. **19**, 1131–1134 (1978).

[48] M. G. Krein, S. N. Saakjan, *Some new results in the theory of resolvents of hermitian operators*, Sov. Math. Dokl. **7**, 1086–1089 (1966).

[49] P. Kurasov and S. T. Kuroda, *Krein's resolvent formula and perturbation theory*, J. Operator Theory **51**, 321–334 (2004).

[50] H. Langer, B. Textorius, *On generalized resolvents and Q-functions of symmetric linear relations (subspaces) in Hilbert space*, Pacific J. Math. **72**, 135–165 (1977).

[51] L. Lanzani and Z. Shen, *On the Robin boundary condition for Laplace's equation in Lipschitz domain*, Comm. Partial Diff. Eqs. **29**, 91–109 (2004).

[52] J. L. Lions, *Espaces d'interpolation et domaines de puissances fractionnaires d'opérateurs*, J. Math. Soc. Japan **14**, 233–241 (1962).

[53] M. M.Malamud, *On a formula of the generalized resolvents of a nondensely defined hermitian operator*, Ukrain. Math. J. **44**, 1522–1547 (1992).

[54] M. M. Malamud and V. I. Mogilevskii, *Krein type formula for canonical resolvents of dual pairs of linear relations*, Methods Funct. Anal. Topology, **8**, No. 4, 72–100 (2002).

[55] M. Marletta, *Eigenvalue problems on exterior domains and Dirichlet to Neumann maps*, J. Comp. Appl. Math. **171**, 367–391 (2004).

[56] V. G. Maz'ja, *Einbettungssätze für Sobolewsche Räume*, Teubner Texte zur Mathematik, Teil 1, 1979; Teil 2, 1980, Teubner, Leipzig.

[57] V. G. Maz'ja, *Sobolev Spaces*, Springer, Berlin, 1985.

[58] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge, 2000.

[59] A. B. Mikhailova, B. S. Pavlov, and L. V. Prokhorov, *Intermediate Hamiltonian via Glazman's splitting and analytic perturbation for meromorphic matrix-functions*, Math. Nachr. **280**, 1376–1416 (2007).

[60] I. Mitrea and M. Mitrea, *Multiple Layer Potentials for Higher Order Elliptic Boundary Value Problems*, preprint, 2008.

[61] S. Nakamura, *A remark on the Dirichlet–Neumann decoupling and the integrated density of states*, J. Funct. Anal. **179**, 136–152 (2001).

[62] G. Nenciu, *Applications of the Krein resolvent formula to the theory of self-adjoint extensions of positive symmetric operators*, J. Operator Theory **10**, 209–218 (1983).

[63] K. Pankrashkin, *Resolvents of self-adjoint extensions with mixed boundary conditions*, Rep. Math. Phys. **58**, 207–221 (2006).

[64] B. Pavlov, *The theory of extensions and explicitly-solvable models*, Russ. Math. Surv. **42:6**, 127–168 (1987).

[65] B. Pavlov, *S-matrix and Dirichlet-to-Neumann operators*, Ch. 6.1.6 in *Scattering: Scattering and Inverse Scattering in Pure and Applied Science*, Vol. 2, R. Pike and P. Sabatier (eds.), Academic Press, San Diego, 2002, pp. 1678–1688.

[66] A. Posilicano, *A Krein-like formula for singular perturbations of self-adjoint operators and applications*, J. Funct. Anal. **183**, 109–147 (2001).

[67] A. Posilicano, *Self-adjoint extensions by additive perturbations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci (5) Vol. II, 1–20 (2003).

[68] A. Posilicano, *Boundary triples and Weyl functions for singular perturbations of self-adjoint operators*, Meth. Funct. Anal. Topology **10**, 57–63 (2004).

[69] A. Posilicano, *Self-adjoint extensions of restrictions*, Operators and Matrices, to appear; arXiv:math-ph/0703078.

[70] V. Ryzhov, *A general boundary value problem and its Weyl function*, Opuscula Math. **27**, 305–331(2007).

[71] V. Ryzhov, *Weyl–Titchmarsh function of an abstract boundary value problem, operator colligations, and linear systems with boundary control*, Complex Anal. Operator Theory, to appear.
[72] Sh. N. Saakjan, *On the theory of the resolvents of a symmetric operator with infinite deficiency indices*, Dokl. Akad. Nauk Arm. SSR **44**, 193–198 (1965). (Russian.)
[73] B. Simon, *Classical boundary conditions as a tool in quantum physics*, Adv. Math. **30**, 268–281 (1978).
[74] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, NJ, 1970.
[75] A. V. Straus, *Generalized resolvents of symmetric operators*, Dokl. Akad. Nauk SSSR **71**, 241–244 (1950). (Russian.)
[76] A. V. Straus, *On the generalized resolvents of a symmetric operator*, Izv. Akad. Nauk SSSR Ser. Math. **18**, 51–86 (1954). (Russian.)
[77] A. V. Straus, *Extensions and generalized resolvents of a non-densely defined symmetric operator*, Math. USSR Izv. **4**, 179–208 (1970).
[78] H. Triebel, *Function spaces in Lipschitz domains and on Lipschitz manifolds. Characteristic functions as pointwise multipliers*, Rev. Mat. Complut. **15**, 475–524 (2002).
[79] E. R. Tsekanovskii and Yu. L. Shmul'yan, *The theory of bi-extensions of operators on rigged Hilbert spaces. Unbounded operator colligations and characteristic functions*, Russ. Math. Surv. **32:5**, 73–131 (1977).
[80] M. Warma, *The Laplacian with general Robin boundary conditions*, Ph.D. Thesis, University of Ulm, 2002.
[81] M. Warma, *The Robin and Wentzell-Robin Laplacians on Lipschitz domains*, Semigroup Forum **73**, 10–30 (2006).
[82] J. Wloka, *Partial Differential Equations*, Cambridge University Press, Cambridge, 1987.
[83] I. Wood, *Maximal $L^p$-regularity for the Laplacian on Lipschitz domains*, Math. Z. **255**, 855–875 (2007).

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
*E-mail address*: `fritz@math.missouri.edu`
*URL*: `http://www.math.missouri.edu/personnel/faculty/gesztesyf.html`

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
*E-mail address*: `marius@math.missouri.edu`
*URL*: `http://www.math.missouri.edu/personnel/faculty/mitream.html`